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ORIGINAL RESEARCH

Generalized rough approximation spaces inspired by cardinality neighborhoods and ideals with application to dengue disease

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Abstract

This article aims to define four new kinds of rough set models based on cardinality neighborhoods and two ideals. The significance of these methods lies in their foundation on ideals, which serve as topological tools. Furthermore, the use of two ideals offers two perspectives instead of just one, thereby reducing the boundary region and increasing the accuracy, which is the primary objective of rough set theory. The concepts of lower and upper approximations based on ideals are presented for the four types. Additionally, we establish essential properties and results for these approximations and construct counterexamples to demonstrate how some of Pawlak's properties have dissipated in the proposed models. The relationships between the current and previous approximations are discussed, and algorithms to classify whether a subset is exact or rough are introduced. Furthermore, we demonstrate how one combination of ideals is applied to address rough paradigms from a topological perspective. Practically, we apply the proposed paradigms to dengue disease management and elucidate two key points: first, our models are distinguished compared to previous ones by retain-

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ing most properties of the original approximation operators proposed by Pawlak; and second, we identify which of the proposed models is better at increasing the accuracy of subsets. In conclusion, we debate the advantages of the suggested models and the motivations behind each type, while also highlighting some of their shortcomings.

Keywords $\mathbb{E}_\xi$-neighborhood · Ideals · Rough sets · Lower and upper approximations · Accuracy criteria

1 Introduction

1.1 Literature review

Pawlak [36, 37] introduced rough set theory as a valuable approach for dealing with uncertain and vague information. This approach confronts uncertainty by splitting data into equivalence classes, aiming to identify both confirmed and possible data obtainable through subsets. It forms a framework for analyzing and understanding data by utilizing approximate sets to describe the properties of the data. This theory has garnered considerable attention and application across various disciplines due to its efficacy in handling imperfect knowledge in data mining, particularly in decision-making analysis [1] and pattern recognition [25, 35]. To accommodate various information systems and address complex practical issues, several extended rough set models have been introduced. These include fuzzy and soft rough sets [33, 49], multi-granulation rough sets [20, 41], and covering rough sets [50, 56].

In Pawlak's standard model, equivalence classes are derived using an equivalence relation, which is a strict condition that limits the efficiency of rough set theory in modeling many real-world problems. Consequently, several models have been proposed that do not require the existence of an equivalence relation. The unit of granular computing (or blocks) used to design these models is the abstract notion of neighborhoods, which describes the geometric characteristic of nearness. The rough set paradigms inspired by these neighborhoods were compared in terms of their ability to achieve the properties of the Pawlak model and their effectiveness in increasing the accuracy measure. These neighborhoods include left and right neighborhoods [51, 52], intersection and union neighborhoods [3, 5, 6], maximal and minimal neighborhoods [8, 20], subset and containment neighborhoods [10, 11], and equal neighborhoods [34], among others. To incorporate more expert opinions, the concept of "ideal" was integrated into earlier rough set models by [28]. Several researchers have since utilized this approach [15, 24, 26] to maximize confirmed knowledge, thereby enhancing the reliability of decision-making methods. It is important to note that many researchers have developed various methods to enhance the properties of rough approximation operators and increase accuracy. These methods include using a family of relations instead of a single relation or employing multiple ideals instead of a single ideal (see [27]).

The relationship between topology and rough set theory was identified early by [47], resulting in the establishment of a topological framework for modeling information systems. Within this framework, the lower approximation corresponds to the topological concept of interior points, and the upper approximation corresponds to the topological concept of closure points. The similarity between these two frameworks has prompted many researchers to revisit set theory through a topological lens, as demonstrated in the studies of [2, 9, 12, 31, 32, 39, 43]. Various methods have been employed to study rough set models within topological spaces, such as using the neighborhood of each point as a subbase of a topology [31] or initiating the topology using the following formula: $\{\mathcal{H} \subseteq \Sigma : \mathcal{G}_{\xi}(\sigma) \subseteq \mathcal{H} \forall \sigma \in \Sigma\}$ [2]. To improve the properties of operators and increase accuracy measures, the concept of ideals was introduced into topological spaces used to study information systems, as investigated in [23, 28, 54]. Additionally, topological generalizations have been employed in modeling set theory and providing descriptions of real-world problems, as seen in [7, 17, 21, 41].

1.2 Gap of research

Upon reviewing the rough set models in the published literature, we note a lack of models inspired by neighborhoods that consider the number of elements related to each other under arbitrary relations. Consequently, we utilize the concept of cardinality neighborhoods as a component of the rough paradigms presented herein. To enhance decision-making confidence by expanding the amount of confirmed information, we improve upon previous rough paradigms by employing two ideals rather than one. The use of two ideals provides two perspectives instead of one, aiming to reduce the boundary region and increase the accuracy degree, which is the primary goal of rough set theory. These results can be routinely generalized to a finite family of ideals. The models presented effectively address challenges related to the cardinality of neighborhoods and demonstrate significant improvements in rough approximation operators, as evidenced by our findings. This work addresses gaps and challenges encountered in practical applications.

1.3 Manuscript's design

The contributions of this work are organized as follows:

(i) In Sect. 2, we review various types of rough neighborhoods and their associated rough set paradigms, as well as some topological methods used to analyze information systems.

(ii) In Sect. 3, we introduce four types of rough set paradigms based on cardinality neighborhoods and two ideals. The motivations behind their introduction will be detailed and their core features will be explored.

(iii) In Sect. 4, we examine the counterparts of the models presented in Subsection 3.4 from a topological standpoint.

(iv) In Sect. 5, we demonstrate how the current framework applies to the modeling of medical information systems, specifically in the context of dengue disease. Also, we compare our models with previous ones in terms of approximation operators and boundary regions.

(v) In Sect. 6, we discuss the advantages and limitations of the current models compared to existing ones.

(vi) In Sect. 7, we summarize the key characteristics and aspects of our models and suggest directions for future research.

2 Preliminaries

This section is dedicated to reviewing several key definitions and results, and to justifying the need for introducing the concepts of cardinal neighborhoods and ideals.

2.1 Traditional approximation space (TAS)

Definition 1 (see, [36]) Let Σ denote a universe, defined as a nonempty finite set. A binary relation ρ on Σ is characterized as a subcollection of $\Sigma \times \Sigma$. The pairing $(\sigma_1, \sigma_2) \in \rho$ is commonly expressed as $\sigma_1 \rho \sigma_2$. A relation ρ on Σ is termed equivalence if it is reflexive, symmetric, and transitive i.e $\sigma \rho \sigma$ for any $\sigma \in \Sigma$, $\sigma_1 \rho \sigma_2 \iff \sigma_2 \rho \sigma_1$, and $\sigma_1 \rho \sigma_3$ when $\sigma_1 \rho \sigma_2$ and $\sigma_2 \rho \sigma_3$. Moreover, ρ is a comparable relation, if it satisfies $\sigma_1 \rho \sigma_2$ or $\sigma_2 \rho \sigma_1$ for all $\sigma_1, \sigma_2 \in \Sigma$.

Definition 2 [37, 38] Let ρ be an equivalence relation on Σ. Supposing $\mathcal{O} \subseteq \Sigma$, so the lower, upper approximations of $\mathcal{O}$ will be represented respectively as:

$$\underline{\rho}(\mathcal{O}) = \cup\{\mathcal{H} \in \Sigma/\rho \mid \mathcal{H} \subseteq \mathcal{O}\}.$$
$$\overline{\rho}(\mathcal{O}) = \cup\{\mathcal{H} \in \Sigma/\rho \mid \mathcal{H} \cap \mathcal{O} \neq \emptyset\},$$

The notation Σ/ρ symbolizes the family comprising all equivalence classes induced by the relation ρ.

The pair (Σ, ρ) is henceforth referred to as an approximation space. A set $\mathcal{O}$ is considered rough rough if, $\overline{\rho}(\mathcal{O})$ and $\underline{\rho}(\mathcal{O})$, are not equal. Conversely, if the upper and lower approximations coincide, the set is termed definable or exact.

The core features of traditional rough set model are outlined in the subsequent proposition.

Proposition 1 [37, 38] *Consider an equivalence relation ρ defined on Σ. For sets $\mathcal{H}, \mathcal{O}$, the next characteristics hold:*

$(L1)$ $\underline{\rho}(\mathcal{H}) \subseteq \mathcal{H}$	$(U1)$ $\mathcal{H} \subseteq \overline{\rho}(\mathcal{H})$
$(L2)$ $\underline{\rho}(\emptyset) = \emptyset$	$(U2)$ $\overline{\rho}(\emptyset) = \emptyset$
$(L3)$ $\underline{\rho}(\Sigma) = \Sigma$	$(U3)$ $\overline{\rho}(\Sigma) = \Sigma$
$(L4)$ $If \mathcal{H} \subseteq \mathcal{O},\ then\ \underline{\rho}(\mathcal{H}) \subseteq \underline{\rho}(\mathcal{O})$	$(U4)$ $If \mathcal{H} \subseteq \mathcal{O},\ then\ \overline{\rho}(\mathcal{H}) \subseteq \overline{\rho}(\mathcal{O})$
$(L5)$ $\underline{\rho}(\mathcal{H} \cap \mathcal{O}) = \underline{\rho}(\mathcal{H}) \cap \underline{\rho}(\mathcal{O})$	$(U5)$ $\overline{\rho}(\mathcal{H} \cap \mathcal{O}) \subseteq \overline{\rho}(\mathcal{H}) \cap \overline{\rho}(\mathcal{O})$
$(L6)$ $\underline{\rho}(\mathcal{H}) \cup \underline{\rho}(\mathcal{O}) \subseteq \underline{\rho}(\mathcal{H} \cup \mathcal{O})$	$(U6)$ $\overline{\rho}(\mathcal{H} \cup \mathcal{O}) = \overline{\rho}(\mathcal{H}) \cup \overline{\rho}(\mathcal{O})$
$(L7)$ $\underline{\rho}(\mathcal{H}^c) = (\overline{\rho}(\mathcal{H}))^c$	$(U7)$ $\overline{\rho}(\mathcal{H}^c) = (\underline{\rho}(\mathcal{H}))^c$
$(L8)$ $\underline{\rho}(\underline{\rho}(\mathcal{H})) = \underline{\rho}(\mathcal{H})$	$(U8)$ $\overline{\rho}(\overline{\rho}(\mathcal{H})) = \overline{\rho}(\mathcal{H})$

(L9) $\underline{\rho}((\underline{\rho}(\mathcal{H}))^c) = (\overline{\rho}(\mathcal{H}))^c$ (U9) $\overline{\rho}((\overline{\rho}(\mathcal{H}))^c) = (\underline{\rho}(\mathcal{H}))^c$

(L10) $\underline{\rho}(\mathcal{O}) = \mathcal{O}, \forall \mathcal{O} \in \Sigma/\rho$ (U10) $\overline{\rho}(\mathcal{O}) = \mathcal{O}, \forall \mathcal{O} \in \Sigma/\rho$

Traditional theory [37, 38] has been extended through various methodologies, with a thorough validation of the properties associated with these extensions. However, some properties have proven to be uncertain. Despite this, obtaining as many of these properties as possible is considered beneficial within these methodologies.

Additionally, rough sets can be numerically characterized using the following two criteria:

Definition 3 [37, 38] Consider an equivalence relation ρ on Σ, the **A**-accuracy and **R**-roughness criteria of $\mathcal{O}$ are determined as:

$$\mathbf{A}(\mathcal{O}) = \frac{\mid \underline{\rho}(\mathcal{O}) \mid}{\mid \overline{\rho}(\mathcal{O}) \mid}, \quad \mid \overline{\rho}(\mathcal{O}) \mid \neq 0.$$
$$\mathbf{R}(\mathcal{O}) = 1 - \mathbf{A}(\mathcal{O}).$$

In many cases, equivalence relations may not be feasible. As a result, the classical approach has been extended by employing weaker relations than full equivalence.

2.2 Sorts of ξ-Neighborhood space

Definition 4 [3, 5, 6, 52, 53] Consider an arbitrary relation ρ on Σ. If $\xi \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle, u, \langle u\rangle\}$, then the ξ-neighborhoods of $\sigma \in \Sigma$, symbolized by $\mathcal{G}_\xi(\sigma)$, are identified as:

(i) $\mathcal{G}_r(\sigma) = \{\eta \in \Sigma : \sigma \ \rho \ \eta\}$.

(ii)

$$\mathcal{G}_{\langle r\rangle}(\sigma) = \begin{cases} \bigcap_{\sigma \in \mathcal{G}_r(\eta)} \mathcal{G}_r(\eta) & : \ \exists\, \mathcal{G}_r(\eta)\ involving\ \sigma \\ \emptyset & : \ Elsewise \end{cases}$$

(iii) $\mathcal{G}_l(\sigma) = \{\eta \in \Sigma : \eta \ \rho \ \sigma\}$.

(iv)

$$\mathcal{G}_{\langle l\rangle}(\sigma) = \begin{cases} \bigcap_{\sigma \in \mathcal{G}_l(\eta)} \mathcal{G}_l(\eta) & : \ \exists\, \mathcal{G}_l(\eta)\ involving\ \sigma \\ \emptyset & : \ Elsewise \end{cases}$$

(v) $\mathcal{G}_i(\sigma) = \mathcal{G}_r(\sigma) \bigcap \mathcal{G}_l(\sigma)$.

(vi) $\mathcal{G}_{\langle i\rangle}(\sigma) = \mathcal{G}_{\langle r\rangle}(\sigma) \bigcap \mathcal{G}_{\langle l\rangle}(\sigma)$.

(vii) $\mathcal{G}_u(\sigma) = \mathcal{G}_r(\sigma) \bigcup \mathcal{G}_l(\sigma)$.

$(viii)$ $\mathcal{G}_{\langle u\rangle}(\sigma) = \mathcal{G}_{\langle r\rangle}(\sigma) \bigcup \mathcal{G}_{\langle l\rangle}(\sigma)$.

Unless otherwise specified, we will assume that ξ belongs to the set $\{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle, u, \langle u\rangle\}$.

Definition 5 [43] Consider a relation ρ on Σ and let ϱ_ξ denote a mapping from Σ to 2^Σ, associating each member $\sigma \in \Sigma$ with its ξ-neighborhood in 2^Σ. Consequently, the triple $(\Sigma, \rho, \varrho_\xi)$ is termed an ξ-neighborhood space, abbreviated as ξ-NS.

The above mentioned sorts of neighborhoods were employed to introduce novel changeability of lower and upper approximations, as well as accuracy (roughness) criteria. To improve the quality of approximations and maximize accuracy, numerous comparisons were conducted among these different types of neighborhoods.

Definition 6 [3, 5, 6, 52, 53] Given a relation ρ on Σ, the lower, and upper approximations of each subset $\mathcal{O}$ regarding to the various kinds of neighborhoods are introduced as:

$$\mathcal{F}_{\mathcal{G}_\xi}(\mathcal{O}) = \{\sigma \in \Sigma : \mathcal{G}_\xi(\sigma) \subseteq \mathcal{O}\},$$
$$\mathcal{F}^{\mathcal{G}_\xi}(\mathcal{O}) = \{\sigma \in \Sigma : \mathcal{G}_\xi(\sigma) \cap \mathcal{O} \neq \emptyset\}.$$

Definition 7 [3, 5, 6, 52, 53] Consider a relation ρ on Σ. The $\mathbf{A}_{\mathcal{G}_\xi}$-accuracy and $\mathrm{R}_{\mathcal{G}_\xi}$-roughness criteria of a nonempty set $\mathcal{O}$ in regard to ρ are represented by:

$$\mathbf{A}_{\mathcal{G}_\xi}(\mathcal{O}) = \frac{\mid \mathcal{F}_{\mathcal{G}_\xi}(\mathcal{O}) \cap \mathcal{O} \mid}{\mid \mathcal{F}^{\mathcal{G}_\xi}(\mathcal{O}) \cup \mathcal{O} \mid}, \text{ and}$$
$$\mathrm{R}_{\mathcal{G}_\xi}(\mathcal{O}) = 1 - \mathbf{A}_{\mathcal{G}_\xi}(\mathcal{O}).$$

Definition 8 (see, [11]) Consider two relations ρ_1 and ρ_2 on Σ such that $\rho_1 \subseteq \rho_2$. The approximations derived from $\mathcal{G}$-neighborhoods show the property of monotonicity in both accuracy, and roughness of any set if $\mathbf{A}_{\mathcal{G}\xi_1}(\mathcal{O}) \geq \mathbf{A}_{\mathcal{G}\xi_2}(\mathcal{O})$ and respectively, $\mathrm{R}_{\mathcal{G}\xi_1}(\mathcal{O}) \leq \mathrm{R}_{\mathcal{G}\xi_2}(\mathcal{O})$.

2.3 Cardinality ξ-neighborhood systems

This section is dedicated to introducing the concept of cardinality neighborhoods for any element in a universe, based on a given binary relation. We will explore their main properties and determine the conditions under which some of these neighborhoods are identical. Illustrative examples are provided to support the derived results and relationships. The study of cardinality neighborhoods aims to enhance the accuracy of approximations.

For any $\xi \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle, u, \langle u\rangle\}$, $|\mathcal{G}_\xi(.)|$ denotes the cardinality of $\mathcal{G}_\xi(.)$.

Definition 9 [14] Consider a relation ρ on Σ. For each ξ, the cardinality neighborhoods of an element σ of Σ (briefly, $\mathbb{E}_\xi(\sigma)$) is defined as:

(i) $\mathbb{E}_r(\sigma) = \{\eta \in \Sigma : |\mathcal{G}_r(\sigma)| = |\mathcal{G}_r(\eta)|\}$.
(ii) $\mathbb{E}_l(\sigma) = \{\eta \in \Sigma : |\mathcal{G}_l(\sigma)| = |\mathcal{G}_l(\eta)|\}$.
(iii) $\mathbb{E}_i(\sigma) = \mathbb{E}_r(\sigma) \cap \mathbb{E}_l(\sigma)$.
(iv) $\mathbb{E}_u(\sigma) = \mathbb{E}_r(\sigma) \cup \mathbb{E}_l(\sigma)$.
(v) $\mathbb{E}_{\langle r\rangle}(\sigma) = \{\eta \in \Sigma : |\mathcal{G}_{\langle r\rangle}(\sigma)| = |\mathcal{G}_{\langle r\rangle}(\eta)|\}$.

(vi) $\mathbb{E}_{\langle l\rangle}(\sigma) = \{\eta \in \Sigma : |\mathcal{G}_{\langle l\rangle}(\sigma)| = |\mathcal{G}_{\langle l\rangle}(\eta)|\}$.
(vii) $\mathbb{E}_{\langle i\rangle}(\sigma) = \mathbb{E}_{\langle r\rangle}(\sigma) \cap \mathbb{E}_{\langle l\rangle}(\sigma)$.
(viii) $\mathbb{E}_{\langle u\rangle}(\sigma) = \mathbb{E}_{\langle r\rangle}(\sigma) \cup \mathbb{E}_{\langle l\rangle}(\sigma)$.

Proposition 2 [14]

(i) $\mathbb{E}_i \subseteq \mathbb{E}_r \cap \mathbb{E}_l \subseteq \mathbb{E}_r \cup \mathbb{E}_l \subseteq \mathbb{E}_u$, *and* $\mathbb{E}_{\langle i\rangle} \subseteq \mathbb{E}_{\langle r\rangle} \cap \mathbb{E}_{\langle l\rangle} \subseteq \mathbb{E}_{\langle r\rangle} \cup \mathbb{E}_{\langle l\rangle} \subseteq \mathbb{E}_{\langle u\rangle}$.
(iii) All $\mathbb{E}_\xi$ *are equal, if* ρ *is a symmetric relation on* Σ.

Proposition 3 [14] *Consider* $(\Sigma, \rho, \varrho_\xi)$ *as an* ξ-NS. *If* $\sigma \in \Sigma$, *then* $\mathbb{E}_\xi(\sigma) \neq \emptyset$ *for each* ξ.

Proposition 4 [14] *Consider* $(\Sigma, \rho, \varrho_\xi)$ *as an* ξ-NS *and* $\sigma \in \Sigma$. *Then,* $\sigma \in \mathbb{E}_\xi(\mathsf{x})$ *iff* $\mathsf{x} \in \mathbb{E}_\xi(\sigma)$, *for each* ξ.

Proposition 5 [14] *Consider* $(\Sigma, \rho, \varrho_\xi)$ *as an* ξ-NS. *If* $\sigma \in \mathbb{E}_\xi(\mathsf{y})$, $\mathsf{y} \in \mathbb{E}_\xi(\mathsf{x})$, *then* $\sigma \in \mathbb{E}_\xi(\mathsf{x})$, *for any* $\xi \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$.

Corollary 1 [14] *Consider* $(\Sigma, \rho, \varrho_\xi)$ *as an* ξ-NS *and* $\sigma \in \Sigma$. *Then,* $\sigma \in \mathbb{E}_\xi(\mathsf{x})$ *iff* $\mathbb{E}_\xi(\sigma) = \mathbb{E}_\xi(\mathsf{x})$, *for any* $\xi \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$.

Corollary 2 [14] *For every* $\xi \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$, *the cardinality neighborhood of elements of* Σ *constitute a partition of* Σ. *That is the relation* ρ *defined by* $\mathsf{x}\rho\sigma \iff \mathsf{x} \in \mathbb{E}_\xi(\sigma)$ *is an equivalence relation for all* $\xi \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$.

Corollary 3 [14] *If* ρ *is a symmetric relation, then the cardinality neighborhood of elements of* Σ *constitute a partition of* Σ *for each* $\xi \in \{u, \langle u\rangle\}$

Proposition 6 [14] $\mathbb{E}_\xi = \mathbb{E}_{\langle\xi\rangle}$ *for* $\xi \in \{r, l, i, u\}$, *if* ρ *is a preorder (i.e., reflexive, transitive) relation on* Σ.

Definition 10 [14] Consider $(\Sigma, \rho, \varrho_\xi)$ as an ξ-NS. Based on cardinality neighborhoods, the $\mathbb{E}_\xi$-lower approximation $\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, and $\mathbb{E}_\xi$-upper approximation $\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$ of a set $\mathcal{O}$, assigned as:

$$\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = \{\sigma \in \Sigma : \mathbb{E}_\xi(\sigma) \subseteq \mathcal{O}\}, \text{ and}$$
$$\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) = \{\sigma \in \Sigma : \mathbb{E}_\xi(\sigma) \cap \mathcal{O} \neq \emptyset\}$$

Definition 11 [14] The $\mathbb{E}_\xi$-boundary, $\mathbb{E}_\xi$-positive, and $\mathbb{E}_\xi$-negative regions of a subset $\mathcal{O}$ within an ξ-NS $(\Sigma, \rho, \varrho_\xi)$ are identified respectively as:

$$\mathbb{B}_{\mathbb{E}_\xi}(\mathcal{O}) = \mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) \setminus \mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$$
$$\mathbb{P}_{\mathbb{E}_\xi}(\mathcal{O}) = \mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}),$$
$$\mathbb{N}_{\mathbb{E}_\xi}(\mathcal{O}) = \Sigma \setminus \mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$$

Furthermore, rough sets defined by cardinality neighborhoods can be numerically characterized using the following two criteria:

Definition 12 [14] The $\mathbb{E}_\xi$-accuracy and $\mathbb{E}_\xi$-roughness criteria of $\mathcal{O} \neq \emptyset$ of an ξ-NS $(\Sigma, \rho, \varrho_\xi)$ are respectively endowed by:

$$\mathbb{A}_{\mathbb{E}_\xi}(\mathcal{O}) = \frac{\mid \mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \mid}{\mid \mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) \mid}, \quad \mid \mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) \mid \neq 0.$$
$$\mathbb{R}_{\mathbb{E}_\xi}(\mathcal{O}) = 1 - \mathbb{A}_{\mathbb{E}_\xi}(\mathcal{O}).$$

Theorem 1 [14] *Consider $(\Sigma, \rho, \varrho_\xi)$ as an ξ-NS. Based on cardinality neighborhoods, the family $\Omega_{\mathbb{E}_\xi} = \{\mathcal{O} \subseteq \Sigma : \forall\, \sigma \in \mathcal{O}, \mathbb{E}_\xi(\sigma) \subseteq \mathcal{O}\}$ constitutes a topology on Σ, for each ξ,*

Lemma 1 [14] *Let $(\Sigma, \rho, \varrho_\xi)$ be an ξ-NS and $\sigma \in \Sigma$. If $\xi \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$, then $\mathbb{E}_\xi(\sigma)$ is $\Omega_{\mathbb{E}\xi}$-open set.*

2.4 Ideals

Definition 13 [30] A non-empty subclass R of 2^Σ is called an ideal on Σ provided that the next conditions are satisfied.

1. The union of any two sets in R is a member of R.
2. Any subset of a member in R is also contained in R.

Definition 14 Let R, T be two ideals on Σ. Then,

1. For an element $\mathcal{O}$ to belong to $\mathrm{R} \cup \mathrm{T}$, it must be contained in either R or T.
2. [27] The smallest collection output by R and T is denoted by $\mathrm{R} \vee \mathrm{T}$ and is specified as follows $\mathrm{R} \vee \mathrm{T} = \{\mathcal{A} \cup \mathcal{B} : \mathcal{A} \in \mathrm{R}, \mathcal{B} \in \mathrm{T}\}$.

Note that It is important to indicate that

1. $\mathrm{R} \cup \mathrm{T}$ may not surely form an ideal.
2. $\mathrm{R} \vee \mathrm{T}$ represents an ideal, as established in Proposition 5.1 of [27].
3. $\mathrm{R} \subseteq \mathrm{R} \cup \mathrm{T} \subseteq \mathrm{R} \vee \mathrm{T}$, and $\mathrm{T} \subseteq \mathrm{R} \cup \mathrm{T} \subseteq \mathrm{R} \vee \mathrm{T}$.

2.5 kinds of approximations described by cardinality neighborhoods and ideals

Definition 15 [15] Consider $(\Sigma, \rho, \varrho_\xi)$ as an ξ-NS and R is an ideal on Σ. Regarding to cardinality neighborhoods and ideals, the duo $({}^{\mathrm{R}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}), {}^{\mathrm{R}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}))$ stands for lower and upper approximations of a set $\mathcal{O}$, respectively, are signified as follows:

$${}^{\mathrm{R}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) = \{\sigma \in \Sigma : \mathbb{E}_\xi(\sigma) \setminus \mathcal{O} \in \mathrm{R}\},$$
$${}^{\mathrm{R}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) = \{\sigma \in \Sigma : \mathbb{E}_\xi(\sigma) \cap \mathcal{O} \notin \mathrm{R}\}$$

Proposition 7 [15] *Let* R, T *be ideals on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$, and $\mathcal{O} \subseteq \Sigma$. If $\mathrm{R} \subseteq \mathrm{T}$, then the following statements hold for each ξ:*

(i) ${}^{\mathrm{R}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})$,

(ii) ${}^{\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$,

Definition 16 [15] Let R be an ideal on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$. Based on cardinality neighborhoods and ideals, the ${}^{\mathrm{R}}\mathbb{E}_\xi$-lower approximation ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, and ${}^{\mathrm{R}}\mathbb{E}_\xi$-upper approximation ${}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$ of any subset $\mathcal{O}$ of Σ are assigned as follows:

$$
\begin{aligned}
{}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) &= {}^{\mathrm{R}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \cap \mathcal{O},\\
{}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) &= {}^{\mathrm{R}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) \cup \mathcal{O}
\end{aligned}
$$

Definition 17 [15] The ${}^{\mathrm{R}}\mathbb{E}_\xi$-boundary, ${}^{\mathrm{R}}\mathbb{E}_\xi$-positive, and ${}^{\mathrm{R}}\mathbb{E}_\xi$-negative regions of a subset $\mathcal{O}$ within an ξ-NS $(\Sigma, \rho, \varrho_\xi)$ with ideal R on Σ are respectively given by

$$
\begin{aligned}
{}^{\mathrm{R}}\mathbb{B}_{\mathbb{E}_\xi}(\mathcal{O}) &= {}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) \setminus {}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})\\
{}^{\mathrm{R}}\mathbb{P}_{\mathbb{E}_\xi}(\mathcal{O}) &= {}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}),\\
{}^{\mathrm{R}}\mathbb{N}_{\mathbb{E}_\xi}(\mathcal{O}) &= \Sigma \setminus {}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}).
\end{aligned}
$$

Definition 18 [15] The ${}^{\mathrm{R}}\mathbb{E}_\xi$-accuracy and ${}^{\mathrm{R}}\mathbb{E}_\xi$-roughness criteria of $\mathcal{O} \neq \emptyset$ of an ξ-NS $(\Sigma, \rho, \varrho_\xi)$ with ideal R on Σ are respectively given by

$$
\begin{aligned}
{}^{\mathrm{R}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) &= \frac{|{}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})|}{|{}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})|}, \; |{}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})| \neq 0.\\
{}^{\mathrm{R}}\mathrm{R}_{\mathbb{E}_\xi}(\mathcal{O}) &= 1 - {}^{\mathrm{R}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}).
\end{aligned}
$$

Theorem 2 [15] *Let* R *be an ideal on an* ξ-NS $(\Sigma, \rho, \zeta_\xi)$. *If* $\mathcal{H}, \mathcal{O} \subseteq \Sigma$, *then for each* ξ *the next statements hold true.*

(i) ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq \mathcal{O} \subseteq {}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$.
(ii) ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\emptyset) = \emptyset$, *and* ${}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\emptyset) = \emptyset$.
(iii) ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\Sigma) = \Sigma$, *and* ${}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\Sigma) = \Sigma$.
(iv) *If* $\mathcal{H} \subseteq \mathcal{O}$, *then* ${}^{\mathrm{R}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{H}) \subseteq {}^{\mathrm{R}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})$ *and* ${}^{\mathrm{R}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{H}) \subseteq {}^{\mathrm{R}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$.
(v) ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}({}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) = {}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, *and* $\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})) = \mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$, *for each* $\xi \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.
(vi) ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}({}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) \subseteq {}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, *and* $\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})) \supseteq \mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$, *for each* $\xi \in \{u, \langle u\rangle\}$.
(vii) *Let* $\eta \in \Sigma$. *Then* ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathbb{E}_\xi(\eta)) = \mathbb{E}_\xi(\eta)$, *for each* $\xi \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.
(viii) *Let* $\eta \in \Sigma$. *Then* ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathbb{E}_\xi(\eta)) \subseteq \mathbb{E}_\xi(\eta)$, *for each* $\xi \in \{u, \langle u\rangle\}$.
(ix) ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) \cap {}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O}) = {}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H} \cap \mathcal{O})$, *and* ${}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H}) \cup {}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{O}) = {}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H} \cup \mathcal{O})$, *for each* ξ.
(x) ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) \cup {}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H} \cup \mathcal{O})$ *and* ${}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H} \cap \mathcal{O}) \subseteq {}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H}) \cap {}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{O})$, *for each* ξ.
(xi) ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}^c) = ({}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}))^c$ *and* ${}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}^c) = ({}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}))^c$.

3 Types of approximations characterized by cardinality neighborhoods and two ideals

This section introduces novel sorts of approximation spaces, which utilize two ideals. These approximations are analyzed using two distinct methods, their properties are examined, and the relationships between these methods are explored.

3.1 First type of rough set paradigms

Definition 19 Consider R, T are ideals on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$. Regarding to cardinality neighborhoods and ideals. The duo $({}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}), {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}))$ stands for lower and upper approximations of a set $\mathcal{O}$, respectively, are signified as follows:

$$
\begin{aligned}
{}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) &= \{\sigma \in \Sigma : \mathbb{E}_\xi(\sigma) \setminus \mathcal{O} \in \mathrm{R} \cup \mathrm{T}\} \\
{}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) &= \{\sigma \in \Sigma : \mathbb{E}_\xi(\sigma) \cap \mathcal{O} \notin \mathrm{R} \cup \mathrm{T}\}
\end{aligned}
$$

By utilizing the operators $({}^{\mathrm{R}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\cdot)$ and ${}^{\mathrm{R}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\cdot))$ furnished in [15], Definition 19 can be reformulated as follows:

Definition 20 Consider R, T are ideals on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$. Regarding to cardinality neighborhoods and ideals, the duo $({}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}), {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}))$ stands for lower and upper approximations of a set $\mathcal{O}$, respectively, are signified as follows:

$$
\begin{aligned}
{}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) &=^{\mathrm{R}} \widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \cup^{\mathrm{T}} \widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \\
{}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) &=^{\mathrm{R}} \widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) \cap^{\mathrm{T}} \widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})
\end{aligned}
$$

Remark 1 The present operators introduced in Definition 20 can be viewed as a real generalization of the operators offered in [15]. Because the current method in Definition 20 coincides with the previous method that described in Definition 3.1 of [15], if one of the following conditions is held:

1. One of the ideal R or T is the empty collection.
2. $\mathrm{T} = \mathrm{R}$.
3. $\mathrm{T} \subseteq \mathrm{R}$ or $\mathrm{R} \subseteq \mathrm{T}$.

Theorem 3 *Let* R, T *be ideals on an* ξ-NS $(\Sigma, \rho, \varrho_\xi)$. *If* $\mathcal{O} \subseteq \Sigma$, *then for each* ξ *the next statements hold true.*

(i) ${}^{\mathrm{R}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})$, *and* ${}^{\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})$.
(ii) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$, *and* ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$.

Proof Utilizing Definition 20, one can prove these statements. □

In the existing models, certain characteristics of Pawlak's paradigm are violated. Some of their deficiencies will be displayed in the following remark:

Remark 2 (i) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\emptyset) \neq \emptyset$, and ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\Sigma) \neq \Sigma$.

Table 1 $\mathbb{E}_\xi$-neighborhoods for members of Σ

	σ_1	σ_2	σ_3	σ_4
$\mathbb{E}_r$	$\{\sigma_1, \sigma_3\}$	$\{\sigma_2\}$	$\{\sigma_1, \sigma_3\}$	$\{\sigma_4\}$
$\mathbb{E}_l$	$\{\sigma_1\}$	$\{\sigma_2\}$	$\{\sigma_3, \sigma_4\}$	$\{\sigma_3, \sigma_4\}$
$\mathbb{E}_i$	$\{\sigma_1\}$	$\{\sigma_2\}$	$\{\sigma_3\}$	$\{\sigma_4\}$
$\mathbb{E}_u$	$\{\sigma_1, \sigma_3\}$	$\{\sigma_2\}$	$\{\sigma_1, \sigma_3, \sigma_4\}$	$\{\sigma_3, \sigma_4\}$
$\mathbb{E}_{\langle r\rangle}$	$\{\sigma_1\}$	$\{\sigma_2, \sigma_4\}$	$\{\sigma_3\}$	$\{\sigma_2, \sigma_4\}$
$\mathbb{E}_{\langle l\rangle}$	$\{\sigma_1\}$	$\{\sigma_2, \sigma_3\}$	$\{\sigma_2, \sigma_3\}$	$\{\sigma_4\}$
$\mathbb{E}_{\langle i\rangle}$	$\{\sigma_1\}$	$\{\sigma_2\}$	$\{\sigma_3\}$	$\{\sigma_4\}$
$\mathbb{E}_{\langle u\rangle}$	$\{\sigma_1\}$	$\{\sigma_2, \sigma_3, \sigma_4\}$	$\{\sigma_2, \sigma_3\}$	$\{\sigma_2, \sigma_4\}$

(ii) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \nsubseteq \mathcal{O} \nsubseteq {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$.
(iii) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}({}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})) \nsupseteq {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})$ for each $\xi \in \{u, \langle u\rangle\}$.
(iv) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}({}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})) \nsubseteq {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$ for each $\xi \in \{u, \langle u\rangle\}$.
(v) If $\eta \in \Sigma$, then ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathbb{E}_\xi(\eta)) \nsubseteq \mathbb{E}_\xi(\eta)$, for each $\xi \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.
(vi) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\xi}(\mathcal{H}) \cap {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\xi}(\mathcal{O}) \neq {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\xi}(\mathcal{H} \cap \mathcal{O})$, for each ξ.
(vii) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\xi}(\mathcal{H}) \cup {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\xi}(\mathcal{O}) \neq {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\xi}(\mathcal{H} \cup \mathcal{O})$, for each ξ.

The subsequent example supports the aforementioned remarks.

Example 1 Consider $\rho = \{(\sigma_1, \sigma_2), (\sigma_2, \sigma_2), (\sigma_2, \sigma_3), (\sigma_3, \sigma_4)\}$ is a binary relation on $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. Then the cardinality neighborhoods of each element of Σ will be calculated in Table 1.

Let $\mathrm{R} = \{\emptyset, \{\sigma_3\}\}$, $\mathrm{T} = \{\emptyset, \{\sigma_1\}\}$ be two ideals. Then, $\mathrm{R} \cup \mathrm{T} = \{\emptyset, \{\sigma_1\}, \{\sigma_3\}\}$. Accordingly, one can be observed that:

(i) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\langle l\rangle}}(\emptyset) = \{\sigma_1\}$, and ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\langle l\rangle}}(\Sigma) = \{\sigma_2, \sigma_3, \sigma_4\}$.
(ii) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_l}(\mathcal{O}) = \{\sigma_1\} \nsubseteq \{\sigma_3\} \nsubseteq \emptyset = {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_l}(\mathcal{O})$, if $\mathcal{O} = \{\sigma_3\}$.
(iii) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\langle u\rangle}}({}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\langle u\rangle}}(\mathcal{O})) = \{\sigma_1\} \nsupseteq \{\sigma_1, \sigma_3\} = {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\langle u\rangle}}(\mathcal{O})$, if $\mathcal{O} = \{\sigma_2\}$.
(iv) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\langle u\rangle}}({}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\langle u\rangle}}(\mathcal{O})) = \{\sigma_2, \sigma_3, \sigma_4\} \nsubseteq \{\sigma_2, \sigma_4\} = {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\langle u\rangle}}(\mathcal{O})$ if $\mathcal{O} = \{\sigma_1, \sigma_3, \sigma_4\}$.
(v) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_l}(\mathbb{E}_l(\eta)) = \{\sigma_1, \sigma_2\} \nsubseteq \{\sigma_2\} = \mathbb{E}_l(\eta)$, if $\eta = \{\sigma_2\}$.
(vi) Let $\mathcal{H} = \{\sigma_1, \sigma_2\}, \mathcal{O} = \{\sigma_2, \sigma_3\}, \mathcal{H} \cap \mathcal{O} = \{\sigma_2\}$, then ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_r}(\mathcal{H}) \cap {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_r}(\mathcal{O}) = \{\sigma_1, \sigma_2, \sigma_3\} \neq \{\sigma_2\} = {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_r}(\mathcal{H} \cap \mathcal{O})$.
(vii) Let $\mathcal{H} = \{\sigma_3, \sigma_4\}, \mathcal{O} = \{\sigma_1, \sigma_4\}, \mathcal{H} \cup \mathcal{O} = \{\sigma_1, \sigma_3, \sigma_4\}$, then ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_r}(\mathcal{H}) \cup {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_r}(\mathcal{O}) = \{\sigma_4\} \neq \{\sigma_1, \sigma_3, \sigma_4\} = {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_r}(\mathcal{H} \cup \mathcal{O})$.

To address these shortcomings while preserving the advantages of the previously introduced rough set model, particularly in enhancing the lower approximation and minimizing the upper approximation, we have structured the following subsection.

3.2 Second type of rough set paradigms

Definition 21 Consider R, T as ideals on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$. With respect to cardinality neighborhoods and ideals, the pair $({}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}), {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}))$ representing the lower and upper approximations of a set $\mathcal{O}$, respectively, are signified as follows:

$$^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = {}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \cup {}^{\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$$
$$^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) = {}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) \cap {}^{\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$$

where $^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, $^{\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, $^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$, $^{\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$ are mentioned in Definition 3.2 in [15].

Definition 22 Consider R, T are ideals on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$. Regarding to cardinality neighborhoods and ideals, the $^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_\xi$-accuracy degree $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O})$ of a set $\mathcal{O}$ is assigned as:

$$^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) = \frac{|^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})|}{|^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})|}, \; |^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})| \neq 0.$$

The following theorem demonstrates that the current models surpass the previous paradigms outlined in [15] by enhancing the lower approximation and minimizing the upper approximation, thereby maximizing accuracy measures.

Theorem 4 *Let* R, T *be ideals on an* ξ-NS $(\Sigma, \rho, \varrho_\xi)$. *If* $\mathcal{O} \subseteq \Sigma$, *then for each* ξ *the next statements hold true.*

(i) $^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, *and* $^{\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$.
(ii) $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$, *and* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$.
(iii) $^{\mathrm{R}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O})$, *and* $^{\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O})$.

Proof Utilizing Definitions 21, 22, one can prove these statements. □

Theorem 5 *Let* R, T *be ideals on an* ξ-NS $(\Sigma, \rho, \varrho_\xi)$. *If* $\mathcal{H}, \mathcal{O} \subseteq \Sigma$, *then for each* ξ *the next statements hold true.*

(i) $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\emptyset) = \emptyset$, *and* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\Sigma) = \Sigma$.
(ii) $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\emptyset) = \emptyset$, *and* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\Sigma) = \Sigma$.
(iii) If $\mathcal{H} \subseteq \mathcal{O}$, *then* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{H}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$ *and* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{H}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$.
(iv) $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq \mathcal{O} \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$.
(v) $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}^c) = ({}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}))^c$ *and* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}^c) = ({}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}))^c$.
(vi) $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}({}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}))$, *and* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}({}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})) = {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$, *for each* $\xi \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.
(vii) $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}({}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, *and* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}({}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})) \supseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$, *for each* $\xi \in \{u, \langle u\rangle\}$.
(viii) Let $\eta \in \Sigma$. *Then* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathbb{E}_\xi(\eta)) = \mathbb{E}_\xi(\eta)$, *for each* $\xi \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.
(ix) Let $\eta \in \Sigma$. *Then* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathbb{E}_\xi(\eta)) \subseteq \mathbb{E}_\xi(\eta)$, *for each* $\xi \in \{u, \langle u\rangle\}$.
(x) $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) \cap {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O}) \supseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H} \cap \mathcal{O})$, *and* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H}) \cup {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H} \cup \mathcal{O})$.
(xi) $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) \cup {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H} \cup \mathcal{O})$ *and* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H} \cap \mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H}) \cap {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{O})$.
(xii) If $\mathcal{O}^c \in \mathrm{R} \cup \mathrm{T}$, *then* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = \mathcal{O}$ *and* $^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}^c) = \mathcal{O}^c$.

Proof According to Definition 21, $(i), (ii), (iii), (iv), (viii), (ix), (x), (xi), (xii)$ are understandable.

(v) ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}^c) = {}^{R}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}^c) \cup {}^{T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}^c)$. From Theorem 2 (xi), ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}^c) = ({}^{R}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}))^c \cup ({}^{T}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}))^c = ({}^{R}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) \cap {}^{T}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}))^c = ({}^{R\diamond T}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}))^c$. By the same manner, ${}^{R\diamond T}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}^c) = ({}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}))^c$.

(vi) Let $\xi \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$. ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = {}^{R}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \cup {}^{T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = {}^{R}\mathcal{F}_{\mathbb{E}_\xi}({}^{R}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) \cup {}^{T}\mathcal{F}_{\mathbb{E}_\xi}({}^{T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) \subseteq {}^{R}\mathcal{F}_{\mathbb{E}_\xi}({}^{R}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \cup {}^{T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) \cup {}^{T}\mathcal{F}_{\mathbb{E}_\xi}({}^{R}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \cup {}^{T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) = {}^{R}\mathcal{F}_{\mathbb{E}_\xi}({}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) \cup {}^{T}\mathcal{F}_{\mathbb{E}_\xi}({}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) = {}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}({}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}))$. Regarding to items $(iii), (iv)$, ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}({}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) \subseteq {}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$. By using (v), ${}^{R\diamond T}\mathcal{F}^{\mathbb{E}_\xi}({}^{R\diamond T}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})) \subseteq {}^{R\diamond T}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$.

(vii) Let $\xi \in \{u, \langle u\rangle\}$. Regarding to items $(iii), (iv)$, ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}({}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) \subseteq {}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, and ${}^{R\diamond T}\mathcal{F}^{\mathbb{E}_\xi}({}^{R\diamond T}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})) \supseteq {}^{R\diamond T}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$.

□

To discuss how the converse of Theorem 4 and properties in some cases of Theorem 5 is not achieved with lower approximations, and to establish similar examples involving upper approximations using property (v), let's first clarify the context and notation involved in these theorems and properties.

Example 2 According to Example 1, we have the following remarks:

(i) Suppose that $\mathcal{O} = \{\sigma_3\}$. If $R = \{\emptyset, \{\sigma_3\}\}$, $T = \{\emptyset, \{\sigma_1\}\}$ are two ideals on Σ, then ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}r}(\mathcal{O}) = \{\sigma_3\}$, and ${}^{R}\mathcal{F}_{\mathbb{E}r}(\mathcal{O}) = \emptyset$. Hence, ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}_r}(\mathcal{O}) \not\subseteq {}^{R}\mathcal{F}_{\mathbb{E}_r}(\mathcal{O})$.

(ii) Suppose that $R = \{\emptyset, \{\sigma_3\}\}$, $T = \{\emptyset, \{\sigma_1\}\}$ are two ideals on Σ. If $\xi = r$, then the converse of (x) for Theorem 5 need not be true. If $\mathcal{H} = \{\sigma_2, \sigma_3, \sigma_4\}$, $\mathcal{O} = \{\sigma_1, \sigma_2, \sigma_4\}$, then ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) = \Sigma$, ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O}) = \Sigma$ and ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H} \cap \mathcal{O}) = \{\sigma_2, \sigma_4\}$. Hence, ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H} \cap \mathcal{O}) \not\supseteq {}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) \cap {}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O})$.

(iii) Suppose that $R = \{\emptyset, \{\sigma_2\}\}$, $T = \{\emptyset, \{\sigma_4\}\}$ are two ideals on Σ. If $\xi = r$, then

1) the converse of (iii) for Theorem 5 is not true. If $\mathcal{H} = \{\sigma_1\}$, $\mathcal{O} = \{\sigma_3\}$, then ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) = \emptyset$, ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O}) = \emptyset$. Hence, ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{H}) \subseteq {}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, but $\mathcal{H} \not\subseteq \mathcal{O}$, and $\mathcal{O} \not\subseteq \mathcal{H}$.
2) the converse of (iv) for Theorem 5 fails in general. If $\mathcal{O} = \{\sigma_1, \sigma_2, \sigma_4\}$, then ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O}) = \{\sigma_2, \sigma_4\}$. Hence, $\mathcal{O} \not\subseteq {}^{R\diamond T}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$.
3) the converse of (xi) for Theorem 5 fails in general. If $\mathcal{H} = \{\sigma_1\}$, $\mathcal{O} = \{\sigma_3\}$, then ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) = \emptyset$, ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O}) = \emptyset$, ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H} \cup \mathcal{O}) = \{\sigma_1, \sigma_3\}$. Hence, ${}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H} \cup \mathcal{O}) \not\subseteq {}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) \cup {}^{R\diamond T}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O})$.

(iv) Suppose that $\mathrm{R} = \{\emptyset, \{\sigma_3\}\}$, $\mathrm{T} = \{\emptyset, \{\sigma_1\}\}$ are two ideals on Σ. If $\xi = \langle u \rangle$, then

1) the converse of (vii) for Theorem 5 need not be true. If $\mathcal{O} = \{\sigma_2, \sigma_3\}$, then ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = \{\sigma_3\}$, and ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}({}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) = \emptyset$. Hence, ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \nsubseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}({}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}))$.
2) the converse of (ix) for Theorem 5 need not be true. Let $\sigma_3 \in \Sigma$. Then $\mathbb{E}_\xi(\sigma_3) = \{\sigma_2, \sigma_3\}$, and ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathbb{E}_\xi(\sigma_3)) = \{\sigma_3\}$. Hence, $\mathbb{E}_\xi(\sigma_3) \nsubseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathbb{E}_\xi(\sigma_3))$.

In the remainder of this subsection, we do the following: 1) expounding on the relationships between the different cases of the proposed rough paradigms, and 2) providing an algorithm to determine whether a subset is classified as ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_\xi$-exact or ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_\xi$-rough.

Proposition 8 *Let* R, T *be ideals on an* ξ*-NS* $(\Sigma, \rho, \varrho_\xi)$. *If* $\mathcal{O} \subseteq \Sigma$, *then*

(i) ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}u}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}r}(\mathcal{O}) \cap {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}r}(\mathcal{O}) \cup {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}i}(\mathcal{O})$.
(ii) ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}i}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}r}(\mathcal{O}) \cap {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}r}(\mathcal{O}) \cup {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}u}(\mathcal{O})$.
(iii) ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle u\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cap {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cup {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle i\rangle}(\mathcal{O})$.
(iv) ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle i\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cap {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cup {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle u\rangle}(\mathcal{O})$.

Proof The proof is warranted by (i) of Proposition 2. □

Corollary 4 *Let* R, T *be ideals on an* ξ*-NS* $(\Sigma, \rho, \varrho_\xi)$. *If* $\mathcal{O} \subseteq \Sigma$, *then*

(i) ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}u}(\mathcal{O}) \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}r}(\mathcal{O}) \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}i}(\mathcal{O})$, *and* ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}u}(\mathcal{O}) \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}l}(\mathcal{O}) \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}i}(\mathcal{O})$.
(ii) ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle u\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle r\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle i\rangle}(\mathcal{O})$, *and* ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle u\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle l\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle i\rangle}(\mathcal{O})$.

Proposition 9 *If* $\mathcal{O}$ *is a nonempty subset of* Σ, *then* $0 \leq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) \leq 1$ *for any* ξ.

Proof Follows by the fact that ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq \mathcal{O} \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$. □

Definition 23 A subset $\mathcal{O}$ is called ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_\xi$-exact, if ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) = 1$. Otherwise, $\mathcal{O}$ is called ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_\xi$-rough.

Eventually, we present Algorithm 1, which determines whether a set is ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_\xi$-exact or ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_\xi$-rough, and subsequently computes its accuracy measure.

Input : The universal set $\varSigma$ under consideration.
Output: Determine whether a subset is ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_{\xi}$-exact or ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_{\xi}$-rough and compute its accuracy.

Insert a relation ρ and two ideal R and T over $\varSigma$ as given by the expert;
Combine $\mathrm{R} \cup \mathrm{T}$;
Select a type of ξ;
for *all* $\sigma \in \varSigma$ **do**
  Compute $\mathcal{G}_{\xi}(\sigma)$
end
for *all* $\sigma \in \varSigma$ **do**
  Compute $\mathbb{E}_{\xi}(\sigma)$
end
for *each subset* $\mathcal{O} \neq \emptyset$ *of* $\varSigma$ **do**
  Compute ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\xi}}(\mathcal{O})$ (by the formula of Definition 19);
  Compute ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{O}) = {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\xi}}(\mathcal{O}) \cap \mathcal{O}$;
  Compute ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\xi}}(\mathcal{O})$ (by the formula of Definition 19);
  Compute ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{O}) = {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\xi}}(\mathcal{O}) \cup \mathcal{O}$;
  if ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{O}) = {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{O})$ **then**
    a subset $\mathcal{O}$ is ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_{\xi}$-exact;
    Print ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}_{\xi}}(\mathcal{O}) = 1$
  else
    a subset $\mathcal{O}$ is ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_{\xi}$-rough;
    Compute ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}_{\xi}}(\mathcal{O}) = \dfrac{|{}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{O})|}{|{}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{O})|}$
  end
end

Algorithm 1: Examination whether a subset is ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_{\xi}$-exact or ${}^{\mathrm{R}\diamond\mathrm{T}}\mathbb{E}_{\xi}$-rough, and compute its accuracy

3.3 Third type of rough set paradigms

This part introduces a new paradigm of rough sets inspired by the concepts of cardinality neighborhoods and two ideals. We demonstrate that this paradigm enhances the lower approximation and reduces the upper approximation of sets compared to previous rough set models.

Definition 24 Consider R, T are ideals on an ξ-NS $(\varSigma, \rho, \varrho_{\xi})$. Regarding to cardinality neighborhoods and ideals, the duo $({}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\xi}}(\mathcal{O}), {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\xi}}(\mathcal{O}))$ stands for lower and upper approximations of a set $\mathcal{O}$, respectively, are signified as follows:

$$
\begin{aligned}
{}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\xi}}(\mathcal{O}) &= \{\sigma \in \varSigma : \mathbb{E}_{\xi}(\sigma) \setminus \mathcal{O} \in \mathrm{R} \vee \mathrm{T}\}, \\
{}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\xi}}(\mathcal{O}) &= \{\sigma \in \varSigma : \mathbb{E}_{\xi}(\sigma) \cap \mathcal{O} \notin \mathrm{R} \vee \mathrm{T}\}.
\end{aligned}
$$

Remark 3 If $\mathrm{T} = \{\emptyset\}$, $\mathrm{T} = \mathrm{R}$ or $\mathrm{T} \subseteq \mathrm{R}$ in Definition 24, then the current method coincides with the previous method described in Definition 3.1 of [15]. Therefore, the present work can be viewed as a real generalization of work presented in [15].

We highlight, in the next findings, the advantages of current models over the models presented in the foregoing subsection.

Theorem 6 *Let* R, T *be ideals on an* ξ*-NS* $(\Sigma, \rho, \varrho_\xi)$*. If* $\mathcal{O} \subseteq \Sigma$*, then for each* ξ *the next statements hold true.*

(i) ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})$.
(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$.

Proof Since $\mathrm{R} \cup \mathrm{T} \subseteq \mathrm{R} \vee \mathrm{T}$, the proof is evident. $\square$

The converse of items of Theorem 6 need not to be true as we note in the next example:

Example 3 Continued in Example 1.

(i) If $\mathcal{O} = \{\sigma_2, \sigma_4\}$, then ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_r}(\mathcal{O}) = \{\sigma_2, \sigma_4\}$ and ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_r}(\mathcal{O}) = \Sigma$. Hence, ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \nsubseteq {}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})$.
(ii) If $\mathcal{O} = \{\sigma_1, \sigma_2, \sigma_3\}$, then ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_r}(\mathcal{O}) = \{\sigma_2\}$ and ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_r}(\mathcal{O}) = \{\sigma_1, \sigma_2, \sigma_3\}$. Hence ${}^{\mathrm{R}\diamond\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) \nsubseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$.

Proposition 10 *Let* R, T *be ideals on an* ξ*-NS* $(\Sigma, \rho, \varrho_\xi)$*. If* $\mathcal{O} \subseteq \Sigma$*, then*

(i) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}u}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}r}(\mathcal{O}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}r}(\mathcal{O}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}i}(\mathcal{O})$.
(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}i}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}r}(\mathcal{O}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}r}(\mathcal{O}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}u}(\mathcal{O})$.
(iii) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\langle u\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\langle i\rangle}(\mathcal{O})$.
(iv) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\langle i\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\langle u\rangle}(\mathcal{O})$.

Proof The proof is warranted by (i) of Proposition 2. $\square$

Proposition 11 *Let* R, T *be ideals on an* ξ*-NS* $(\Sigma, \rho, \varrho_\xi)$*. If* $\mathcal{O} \subseteq \Sigma$*, then*

(i) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\xi}(\mathcal{H}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\xi}(\mathcal{O}) = {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}\xi}(\mathcal{H} \cap \mathcal{O})$*, for each* ξ.
(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\xi}(\mathcal{H}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\xi}(\mathcal{O}) = {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}\xi}(\mathcal{H} \cup \mathcal{O})$*, for each* ξ.

Proof The proof is obvious, since $\mathrm{R} \vee \mathrm{T}$ is ideal. $\square$

In the available models, specific features of Pawlak's model are disrupted. Some of these inadequacies are accentuated in the following remark:

Remark 4 (i) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\emptyset) \neq \emptyset$, and ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\Sigma) \neq \Sigma$.
(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \nsubseteq \mathcal{O} \nsubseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$.
(iii) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}({}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})) \nsupseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})$ for each $\xi \in \{u, \langle u\rangle\}$.
(iv) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}({}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})) \nsubseteq {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$ for each $\xi \in \{u, \langle u\rangle\}$.
(v) If $\eta \in \Sigma$, then ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{E}_\xi(\eta)) \nsubseteq \mathcal{E}_\xi(\eta)$, for each $\xi \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

Example 4 Continued in Example 1. Let $\mathrm{R} = \{\emptyset, \{\sigma_3\}\}$, $\mathrm{T} = \{\emptyset, \{\sigma_1\}\}$ be two ideals. Then, $\mathrm{R} \vee \mathrm{T} = \{\emptyset, \{\sigma_1\}, \{\sigma_3\}, \{\sigma_1, \sigma_3\}\}$. Accordingly, one can be observed that:

(i) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\langle i\rangle}}(\emptyset) = \{\sigma_1, \sigma_3\}$, and ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\langle i\rangle}}(\Sigma) = \{\sigma_2, \sigma_4\}$.
(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_l}(\mathcal{O}) = \{\sigma_1\} \nsubseteq \{\sigma_3\} \nsubseteq \emptyset = {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_l}(\mathcal{O})$, if $\mathcal{O} = \{\sigma_3\}$.
(iii) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\langle u\rangle}}({}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\langle u\rangle}}(\mathcal{O})) = \{\sigma_1\} \nsupseteq \{\sigma_1, \sigma_3\} = {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_{\langle u\rangle}}(\mathcal{O})$, if $\mathcal{O} = \{\sigma_2\}$.
(iv) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\langle u\rangle}}({}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\langle u\rangle}}(\mathcal{O})) = \{\sigma_2, \sigma_3, \sigma_4\} \nsubseteq \{\sigma_2, \sigma_4\} = {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_{\langle u\rangle}}(\mathcal{O})$ if $\mathcal{O} = \{\sigma_1, \sigma_3, \sigma_4\}$.
(v) ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_r}(\mathcal{E}_r(\eta)) = \{\sigma_1, \sigma_2, \sigma_3\} \nsubseteq \{\sigma_2\} = \mathcal{E}_r(\eta)$, if $\eta = \{\sigma_2\}$.

To contend these imperfectness while conserving the advantages of the foregoing rough set model (introduced in this subsection), especially in terms of augmenting lower approximation and decreasing upper approximation, we form up the next subsection.

3.4 Fourth type of rough set paradigms

Definition 25 Consider R, T are ideals on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$. Regarding to cardinality neighborhoods and ideals, the duo $({}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}), {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}))$ stands for lower and upper approximations of a set $\mathcal{O}$, respectively, are signified as follows:

$$ {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = \{\sigma_1 \in \Sigma : \mathbb{E}_\xi(\sigma_1) \setminus \mathcal{O} \in \mathrm{R} \vee \mathrm{T}\} \cap \mathcal{O}, $$
$$ {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) = \{\sigma_1 \in \Sigma : \mathbb{E}_\xi(\sigma_1) \cap \mathcal{O} \notin \mathrm{R} \vee \mathrm{T}\} \cup \mathcal{O}. $$

Definition 26 Consider R, T are ideals on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$. Regarding to cardinality neighborhoods and ideals, the ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-accuracy degree ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O})$ of a set $\mathcal{O}$ is assigned as:

$$ {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) = \frac{|{}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})|}{|{}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})|}, \ |{}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})| \neq 0. $$

Firstly, we illustrate that the current rough set models are better that their counterparts introduced in Subsection 3.2 in terms of enlarging the lower approximation and minifying the upper approximation.

Proposition 12 *Let* R, T *be ideals on an* ξ-NS $(\Sigma, \rho, \varrho_\xi)$. *If* $\mathcal{O} \subseteq \Sigma$, *then for each* ξ *the next statements hold true.*

(i) ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$.
(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$.

Proof It follows from the fact that $\mathrm{R} \cup \mathrm{T} \subseteq \mathrm{R} \vee \mathrm{T}$. □

Corollary 5 *Let* R, T *be ideals on an* ξ-NS $(\Sigma, \rho, \varrho_\xi)$. *If* $\mathcal{O} \subseteq \Sigma$, *then*

$$ {}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}). $$

The subsequent instance demonstrates that the converse of the above proposition and corollary is incorrect in general.

Example 5 Let ρ and T as given in Example 1. Put R $= \{\emptyset, \{\sigma_4\}\}$ and let $\mathcal{O} = \{\sigma_3\}$. Then, ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{F}_{\mathbb{E}_u}(\mathcal{O}) = \emptyset$, whereas ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_u}(\mathcal{O}) = \{\sigma_3\}$.

We now proceed to examine the properties of ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}()$, ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}()$ for any set, as outlined in the following results.

Theorem 7 *Let* R, T *be ideals on an* ξ*-NS* $(\Sigma, \rho, \varrho_\xi)$. *If* $\mathcal{H}, \mathcal{O} \subseteq \Sigma$, *then for each* ξ *the next statements hold true.*

(i) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\emptyset) = \emptyset$, *and* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\Sigma) = \Sigma$.

(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\emptyset) = \emptyset$, *and* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\Sigma) = \Sigma$.

(iii) *If* $\mathcal{H} \subseteq \mathcal{O}$, *then* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{H}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$ *and* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{H}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$.

(iv) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq \mathcal{O} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$.

(v) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}^c) = ({}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}))^c$ *and* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}^c) = ({}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}))^c$.

(vi) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}({}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}))$, *and* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}({}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})) = {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$, *for each* $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

(vii) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}({}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$, *and* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}({}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})) \supseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$, *for each* $\sigma \in \{u, \langle u\rangle\}$.

(viii) *Let* $\eta \in \Sigma$. *Then* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{E}_\xi(\eta)) = \mathcal{E}_\xi(\eta)$, *for each* $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

(ix) *Let* $\eta \in \Sigma$. *Then* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathbb{E}_\xi(\eta)) \subseteq \mathbb{E}_\xi(\eta)$, *for each* $\sigma \in \{u, \langle u\rangle\}$.

(x) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O}) = {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H} \cap \mathcal{O})$, *and* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{O}) = {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H} \cup \mathcal{O})$, *for each* ξ.

(xi) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\xi}(\mathcal{H} \cup \mathcal{O})$ *and* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H} \cap \mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{H}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\xi}(\mathcal{O})$, *for each* ξ.

Proof Direct to prove. □

Proposition 13 *Let* R *be an ideal on an* ξ*-NS* $(\Sigma, \rho, \varrho_\xi)$. *If* $\mathcal{O} \subseteq \Sigma$, *then*

(i) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}u}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}r}(\mathcal{O}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}r}(\mathcal{O}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}i}(\mathcal{O})$.

(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}i}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}r}(\mathcal{O}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}r}(\mathcal{O}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}l}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}u}(\mathcal{O})$.

(iii) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle u\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}\langle i\rangle}(\mathcal{O})$.

(iv) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle i\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cap {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle r\rangle}(\mathcal{O}) \cup {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle l\rangle}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}\langle u\rangle}(\mathcal{O})$.

Proof The proof is warranted by (i) of Proposition 2. □

Corollary 6 *Let* R *be an ideal on an* ξ*-NS* $(\Sigma, \rho, \varrho_\xi)$. *If* $\mathcal{O} \subseteq \Sigma$, *then*

(i) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}u}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}r}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}i}(\mathcal{O})$, *and* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}u}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}l}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}i}(\mathcal{O})$.

(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle u\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle r\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle i\rangle}(\mathcal{O})$, *and* ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle u\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle l\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}\langle i\rangle}(\mathcal{O})$.

Proposition 14 *If $\mathcal{O}$ is a nonempty subset of Σ, then* $0 \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) \leq 1$ *for any* σ.

Proof Follows by the fact that ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) \subseteq \mathcal{O} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$. □

Definition 27 A subset $\mathcal{O}$ is called ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-exact, if ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) = 1$. Otherwise, $\mathcal{O}$ is called ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-rough.

Eventually, we furnish Algorithm 2 to specify if a set is ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-exact or ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-rough and then compute its accuracy measure.

Input : The universal set Σ under consideration.
Output: Determine whether a subset is ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-exact or ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-rough and compute its accuracy.

Insert a relation ρ and two ideal R and T over Σ as given by the expert;
Combine $\mathrm{R} \vee \mathrm{T}$;
Select a type of ξ;
for *all* $\sigma \in \Sigma$ **do**
  Compute $\mathcal{G}_\xi(\sigma)$
end
for *all* $\sigma \in \Sigma$ **do**
  Compute $\mathbb{E}_\xi(\sigma)$
end
for *each subset* $\mathcal{O} \neq \emptyset$ *of* Σ **do**
  Compute ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})$ (by the formula of Definition 24);
  Compute ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O}) \cap \mathcal{O}$;
  Compute ${}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O})$ (by the formula of Definition 24);
  Compute ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O}) = {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}^{\mathbb{E}_\xi}(\mathcal{O}) \cup \mathcal{O}$;
  if ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O}) = {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$ **then**
    a subset $\mathcal{O}$ is ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-exact;
    Print ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) = 1$
  else
    a subset $\mathcal{O}$ is ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-rough;
    Compute ${}^{\mathrm{R}\diamond\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) = \dfrac{|{}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})|}{|{}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})|}$
  end
end

Algorithm 2: Examination whether a subset is ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-exact or ${}^{\mathrm{R}\vee\mathrm{T}}\mathbb{E}_\xi$-rough, and compute its accuracy

4 Various topologies induced by cardinality neighborhoods and two ideals

In this section, we employ cardinal neighborhoods and two ideals to get diverse topologies that are finer than those previously produced by cardinal neighborhoods in terms of one ideal, as described in [14], for any given relation.

Theorem 8 *Let* R, T *be ideals on an* ξ-NS $(\Sigma, \rho, \varrho_\xi)$. *For each* ξ, *the family* ${}^{R\vee T}\Omega_{\mathbb{E}_\xi} = \{\mathcal{O} \subseteq \Sigma\colon \forall\, \sigma \in \mathcal{O}, \mathbb{E}_\xi(\sigma)\backslash\mathcal{O} \in R \vee T\}$ *constitutes a topology on* Σ.

Proof Firstly, suppose $\mathcal{O}_\iota \in {}^{R\vee T}\Omega_{\mathbb{E}_\xi}$, for each $\iota \in \Delta$. Let $\sigma \in \cup_{\iota\in\Delta}\mathcal{O}_\iota$, then there is $\iota_0 \in \Delta$ s.t. $\sigma \in \mathcal{O}_{\iota_0}$ and $\mathbb{E}_\xi(\sigma)\backslash\mathcal{O}_{\iota_0} \in R \vee T$. Since $\mathcal{O}_{\iota_0} \subseteq \cup_{\iota\in\Delta}\mathcal{O}_\iota$. Therefore, $\mathbb{E}_\xi(\sigma)\backslash(\cup_{\iota\in\Delta}\mathcal{O}_\iota) \in R \vee T$, this means that $\cup_{\iota\in\Delta}\mathcal{O}_\iota \in {}^{R\vee T}\Omega_{\mathbb{E}_\xi}$.
Secondly, let $\mathcal{O}_1, \mathcal{O}_2$ be elements of ${}^{R\vee T}\Omega_{\mathbb{E}_\xi}$ and σ belongs to the intersection of $\mathcal{O}_1$ and $\mathcal{O}_2$. Then $\mathbb{E}_\xi(\sigma)\backslash\mathcal{O}_1 \in R \vee T$ and $\mathbb{E}_\xi(\sigma)\backslash\mathcal{O}_2 \in R \vee T$. Hence, $\mathbb{E}_\xi(\sigma)\backslash[\mathcal{O}_1 \cap \mathcal{O}_2] \in R \vee T$. This means that $\mathcal{O}_1 \cap \mathcal{O}_2 \in {}^{R\vee T}\Omega_{\mathbb{E}_\xi}$.
Finally, it is evident that $\emptyset, \Sigma \in {}^{R\vee T}\Omega_{\mathbb{E}_\xi}$, for each ξ. Consequently, ${}^{R\vee T}\Omega_{\mathbb{E}_\xi}$ is a topology on Σ. ◻

If $\mathcal{O} \in {}^{R\vee T}\Omega_{\mathbb{E}_\xi}$, then $\mathcal{O}$ is said to be ${}^{R\vee T}\Omega_{\mathbb{E}_\xi}$-open set and its complement is named a ${}^{R\vee T}\perp_{\mathbb{E}_\xi}$-closed set, where ${}^{R\vee T}\perp_{\mathbb{E}_\xi} = \{F : F^c \in {}^{R\vee T}\Omega_{\mathbb{E}_\xi}\}$.

Example 6 Continuing from Example 1.
If $R = \{\emptyset, \{\sigma_3\}\}$, then

$$\begin{aligned}
{}^{R}\Omega_{\mathbb{E}_r} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_4\}, \{\sigma_1,\sigma_2\}, \{\sigma_1,\sigma_3\}, \{\sigma_1,\sigma_4\}, \{\sigma_2,\sigma_4\},\\
&\quad \{\sigma_1,\sigma_2,\sigma_3\}, \{\sigma_1,\sigma_2,\sigma_4\}, \{\sigma_1,\sigma_3,\sigma_4\}\}.\\
{}^{R}\Omega_{\mathbb{E}_l} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_4\}, \{\sigma_1,\sigma_2\}, \{\sigma_3,\sigma_4\}, \{\sigma_1,\sigma_4\},\\
&\quad \{\sigma_2,\sigma_4\}, \{\sigma_2,\sigma_3,\sigma_4\}, \{\sigma_1,\sigma_2,\sigma_4\}, \{\sigma_1,\sigma_3,\sigma_4\}\}.\\
{}^{R}\Omega_{\mathbb{E}_i} &= 2^{\Sigma}.\\
{}^{R}\Omega_{\mathbb{E}_u} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_4\}, \{\sigma_1,\sigma_2\}, \{\sigma_1,\sigma_4\}, \{\sigma_2,\sigma_4\}, \{\sigma_1,\sigma_2,\sigma_4\}, \{\sigma_1,\sigma_3,\sigma_4\}\}.\\
{}^{R}\Omega_{\mathbb{E}\langle r\rangle} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_3\}, \{\sigma_1,\sigma_3\}, \{\sigma_2,\sigma_4\}, \{\sigma_1,\sigma_2,\sigma_4\}, \{\sigma_2,\sigma_3,\sigma_4\}\}.\\
{}^{R}\Omega_{\mathbb{E}\langle l\rangle} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_4\}, \{\sigma_1,\sigma_2\}, \{\sigma_1,\sigma_4\}, \{\sigma_2,\sigma_3\}, \{\sigma_2,\sigma_4\},\\
&\quad \{\sigma_1,\sigma_2,\sigma_3\}, \{\sigma_1,\sigma_2,\sigma_4\}, \{\sigma_2,\sigma_3,\sigma_4\}\}.\\
{}^{R}\Omega_{\mathbb{E}\langle i\rangle} &= 2^{\Sigma}.\\
{}^{R}\Omega_{\mathbb{E}\langle u\rangle} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2,\sigma_4\}, \{\sigma_1,\sigma_2,\sigma_4\}, \{\sigma_2,\sigma_3,\sigma_4\}\}.
\end{aligned}$$

If $T = \{\emptyset, \{\sigma_1\}\}$, then

$$\begin{aligned}
{}^{T}\Omega_{\mathbb{E}_r} &= \{\emptyset, \Sigma, \{\sigma_2\}, \{\sigma_3\}, \{\sigma_4\}, \{\sigma_1,\sigma_3\}, \{\sigma_2,\sigma_3\}, \{\sigma_2,\sigma_4\}, \{\sigma_3,\sigma_4\},\\
&\quad \{\sigma_1,\sigma_2,\sigma_3\}, \{\sigma_2,\sigma_3,\sigma_4\}, \{\sigma_1,\sigma_3,\sigma_4\}\}.\\
{}^{T}\Omega_{\mathbb{E}_l} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_1,\sigma_2\}, \{\sigma_3,\sigma_4\}, \{\sigma_2,\sigma_3,\sigma_4\}, \{\sigma_1,\sigma_3,\sigma_4\}\}.\\
{}^{T}\Omega_{\mathbb{E}_i} &= 2^{\Sigma}.\\
{}^{T}\Omega_{\mathbb{E}_u} &= \{\emptyset, \Sigma, \{\sigma_2\}, \{\sigma_3,\sigma_4\}, \{\sigma_2,\sigma_3,\sigma_4\}, \{\sigma_1,\sigma_3,\sigma_4\}\}.\\
{}^{T}\Omega_{\mathbb{E}\langle r\rangle} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_3\}, \{\sigma_1,\sigma_3\}, \{\sigma_2,\sigma_4\}, \{\sigma_1,\sigma_2,\sigma_4\}, \{\sigma_2,\sigma_3,\sigma_4\}\}.\\
{}^{T}\Omega_{\mathbb{E}\langle l\rangle} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_4\}, \{\sigma_1,\sigma_4\}, \{\sigma_2,\sigma_3\}, \{\sigma_1,\sigma_2,\sigma_3\}, \{\sigma_2,\sigma_3,\sigma_4\}\}.\\
{}^{T}\Omega_{\mathbb{E}\langle i\rangle} &= 2^{\Sigma}.\\
{}^{T}\Omega_{\mathbb{E}\langle u\rangle} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2,\sigma_3,\sigma_4\}\}.
\end{aligned}$$

If $\mathrm{R} \vee \mathrm{T} = \{\emptyset, \{\sigma_1, \{\sigma_3, \{\sigma_1, \sigma_3\}\}$, then

$$\begin{aligned}
{}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_r} &= 2^{\Sigma}.\\
{}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_l} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_4\}, \{\sigma_1, \sigma_2\}, \{\sigma_3, \sigma_4\}, \{\sigma_1, \sigma_4\},\\
&\quad \{\sigma_2, \sigma_4\}, \{\sigma_2, \sigma_3, \sigma_4\}, \{\sigma_1, \sigma_2, \sigma_4\}, \{\sigma_1, \sigma_3, \sigma_4\}\}.\\
{}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_i} &= 2^{\Sigma}.\\
{}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_u} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_4\}, \{\sigma_1, \sigma_2\}, \{\sigma_1, \sigma_4\}, \{\sigma_2, \sigma_4\},\\
&\quad \{\sigma_3, \sigma_4\}, \{\sigma_1, \sigma_2, \sigma_4\}, \{\sigma_1, \sigma_3, \sigma_4\}, \{\sigma_2, \sigma_3, \sigma_4\}\}.\\
{}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}\langle r\rangle} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_3\}, \{\sigma_1, \sigma_3\}, \{\sigma_2, \sigma_4\},\\
&\quad \{\sigma_1, \sigma_2, \sigma_4\}, \{\sigma_2, \sigma_3, \sigma_4\}\}.\\
{}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}\langle l\rangle} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_4\}, \{\sigma_1, \sigma_2\}, \{\sigma_1, \sigma_4\},\\
&\quad \{\sigma_2, \sigma_3\}, \{\sigma_2, \sigma_4\}, \{\sigma_1, \sigma_2, \sigma_3\}, \{\sigma_1, \sigma_2, \sigma_4\}, \{\sigma_2, \sigma_3, \sigma_4\}\}.\\
{}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}\langle i\rangle} &= 2^{\Sigma}.\\
{}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}\langle u\rangle} &= \{\emptyset, \Sigma, \{\sigma_1\}, \{\sigma_2, \sigma_4\}, \{\sigma_1, \sigma_2, \sigma_4\}, \{\sigma_2, \sigma_3, \sigma_4\}\}.
\end{aligned}$$

Proposition 15 *Let* R *be an ideal on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$. Then*

(i) For each ξ, ${}^{\mathrm{R}}\Omega_{\mathbb{E}_\xi} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_\xi}$, and ${}^{\mathrm{T}}\Omega_{\mathbb{E}_\xi} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_\xi}$.
(ii) If ρ is preorder relation, then ${}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_\xi} = {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_{\langle\xi\rangle}}$, for $\xi \in \{r, l, i, u\}$.

Proof (i) Obviously, ${}^{\mathrm{R}}\Omega_{\mathbb{E}_\xi} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_\xi}$, and ${}^{\mathrm{T}}\Omega_{\mathbb{E}_\xi} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_\xi}$, according to the fact that $\mathrm{R} \subseteq \mathrm{R} \vee \mathrm{T}$, and $\mathrm{T} \subseteq \mathrm{R} \vee \mathrm{T}$.
(ii) Obvious by Proposition 6.

□

The converse of Proposition 15 need not to be true, as we see in the following example:

Example 7 Continuing from Example 6. Let $\mathrm{R} = \{\emptyset, \{\sigma_3\}\}$, $\mathrm{T} = \{\emptyset, \{\sigma_1\}\}$, and $\sigma = r$. Then, ${}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_r} = 2^{\Sigma} \nsubseteq {}^{\mathrm{R}}\Omega_{\mathbb{E}_r}$ and ${}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_r} \nsubseteq {}^{\mathrm{T}}\Omega_{\mathbb{E}_r}$.

Theorem 9 *The following properties hold for the topologies generated by cardinality neighborhoods and two ideals:*

(i) ${}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_u} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_r} \cap {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_l} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_r} \cup {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_l} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_i}$.
(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_{\langle u\rangle}} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_{\langle r\rangle}} \cap {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_{\langle l\rangle}} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_{\langle r\rangle}} \cup {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_{\langle l\rangle}} \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_{\langle i\rangle}}$.

Proof A direct result of (i) of Proposition 2. □

Subsequently, we will construct multiform of rough approximations using the topologies generated from cardinal neighborhoods and two ideals. Additionally, we will discuss some of their properties.

Definition 28 Let ${}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_\xi}$ be the topologies generated by cardinality neighborhoods and two ideals, for each ξ. Then, the lower and upper approximations, of any set $\mathcal{O}$, are endowed by:

$\underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})={}^{R\vee T}int_{\mathbb{E}_{\xi}}(\mathcal{O}), \overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})={}^{R\vee T}cl_{\mathbb{E}_{\xi}}(\mathcal{O})$, (where ${}^{R\vee T}int_{\mathbb{E}_{\xi}}(\mathcal{O}), {}^{R\vee T}cl_{\mathbb{E}_{\xi}}(\mathcal{O})$ represent interior, closure respectively of a set $\mathcal{O}$ with respect the topology ${}^{R\vee T}\Omega_{\mathbb{E}_{\xi}}$). Additionally, the accuracy criteria of $\mathcal{O}$ is assigned as: ${}^{R\vee T}\varphi_{\lambda_{\xi}}(\mathcal{O}) = \frac{|\underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})|}{|\overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})|}$, $|\ \overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})\ |\neq 0$.

It is noticeable that $0 \leq {}^{R\vee T}\varphi_{\lambda_{\xi}} \leq 1$. If ${}^{R\vee T}\varphi_{\lambda_{\xi}}(\mathcal{O}) = 1$, then $\mathcal{O}$ is specified to as an ${}^{R\vee T}\mathbb{E}_{\xi}$-exact set. Otherwise, $\mathcal{O}$ is designated an ${}^{R\vee T}\mathbb{E}_{\xi}$-rough set.
Regarding Definition 28, the following outcomes can be certified utilizing the characteristics of interior and closure topological operators. It is remarkable that specific properties, which are absent in the ${}^{R\vee T}\widetilde{\mathcal{F}}_{\mathbb{E}_{\xi}}$-, ${}^{R\vee T}\widetilde{\mathcal{F}}^{\mathbb{E}_{\xi}}$-approximations remain valid for the $\underline{{}^{R\vee T}\lambda_{\xi}}$-, $\overline{{}^{R\vee T}\lambda_{\xi}}$-approximations.

Theorem 10 *For each ξ, suppose that ${}^{R\vee T}\Omega_{\mathbb{E}_{\xi}}$ is a topology generated by cardinality neighborhood and ideal. If $\mathcal{H}, \mathcal{O} \subseteq \Sigma$, then the next statements are valid:*

- *(i)* $\underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O}) \subseteq \mathcal{O}$.
- *(ii)* $\underline{{}^{R\vee T}\lambda_{\xi}}(\emptyset) = \emptyset$.
- *(iii)* $\underline{{}^{R\vee T}\lambda_{\xi}}(\Sigma) = \Sigma$.
- *(iv)* *If $\mathcal{H} \subseteq \mathcal{O}$, then* $\underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{H}) \subseteq \underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})$.
- *(v)* $\underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{H} \cap \mathcal{O}) = \underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{H}) \cap \underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})$.
- *(vi)* $\underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O}^{c}) = (\overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O}))^{c}$.
- *(vii)* $\underline{{}^{R\vee T}\lambda_{\xi}}(\underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})) = \underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})$.

Proof Straightforward from the properties of an interior topological operator. □

Corollary 7 *For each ξ, suppose that ${}^{R\vee T}\Omega_{\mathbb{E}_{\xi}}$ is a topology generated by cardinality neighborhood and ideal. Then $\underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{H}) \cup \underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O}) \subseteq \underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{H} \cup \mathcal{O})$ for any $\mathcal{H}, \mathcal{O} \subseteq \Sigma$.*

Theorem 11 *For each ξ, suppose that ${}^{R\vee T}\Omega_{\mathbb{E}_{\xi}}$ is a topology generated by cardinality neighborhood and ideal. If $\mathcal{H}, \mathcal{O} \subseteq \Sigma$, then the next statements are valid:*

- *(i)* $\mathcal{O} \subseteq \overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})$.
- *(ii)* $\overline{{}^{R\vee T}\lambda_{\xi}}(\emptyset) = \emptyset$.
- *(iii)* $\overline{{}^{R\vee T}\lambda_{\xi}}(\Sigma) = \Sigma$.
- *(iv)* *If $\mathcal{H} \subseteq \mathcal{O}$, then* $\overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{H}) \subseteq^{R\vee T} \overline{\lambda_{\xi}}(\mathcal{O})$.
- *(v)* $\overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{H} \cup \mathcal{O}) = \overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{H}) \cup \overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})$.
- *(vi)* $\overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O}^{c}) = (\underline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O}))^{c}$.
- *(vii)* $\overline{{}^{R\vee T}\lambda_{\xi}}(\overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})) = \overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})$.

Proof From the properties of a closure topological operator, the proof is understandable. □

Corollary 8 *Let ${}^{R\vee T}\Omega_{\mathbb{E}_{\xi}}$ be the topologies generated by cardinality neighborhoods and two ideals. Then $\overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{H} \cap \mathcal{O}) \subseteq \overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{H}) \cap \overline{{}^{R\vee T}\lambda_{\xi}}(\mathcal{O})$ for any $\mathcal{H}, \mathcal{O} \subseteq \Sigma$.*

Proposition 16 *If $\mathcal{O}$ is a nonempty subset of Σ, then* $0 \leq {}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_\xi}(\mathcal{O}) \leq 1$ *for any* ξ.

Proposition 17 *Let* ${}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_\xi}$ *be the topologies generated by cardinality neighborhoods and two ideals, then* ${}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_\xi}(\Sigma) = 1$.

Proposition 18 *Let* ${}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_\xi}$ *be the topologies generated by cardinality neighborhoods and two ideals. If $\mathcal{O}$ is a nonempty subset of Σ, then*

(i) $\underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_u}(\mathcal{O}) \subseteq \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_r}(\mathcal{O}) \cap \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_l}(\mathcal{O}) \subseteq \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_r}(\mathcal{O}) \cup \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_l}(\mathcal{O}) \subseteq \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_i}(\mathcal{O})$.
(ii) $\overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_i}(\mathcal{O}) \subseteq \overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_r}(\mathcal{O}) \cap \overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_l}(\mathcal{O}) \subseteq \overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_r}(\mathcal{O}) \cup \overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_l}(\mathcal{O}) \subseteq \overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_u}(\mathcal{O})$.
(iii) $\underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle u\rangle}}(\mathcal{O}) \subseteq \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle r\rangle}}(\mathcal{O}) \cap \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle l\rangle}}(\mathcal{O}) \subseteq \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle r\rangle}}(\mathcal{O}) \cup \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle l\rangle}}(\mathcal{O}) \subseteq \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle i\rangle}}(\mathcal{O})$.
(iv) $\overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle i\rangle}}(\mathcal{O}) \subseteq \overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle r\rangle}}(\mathcal{O}) \cap \overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle l\rangle}}(\mathcal{O}) \subseteq \overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle r\rangle}}(\mathcal{O}) \cup \overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle l\rangle}}(\mathcal{O}) \subseteq \overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_{\langle u\rangle}}(\mathcal{O})$.

Corollary 9 *Let* ${}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_\xi}$ *be the topologies generated by cardinality neighborhoods and ideals. If $\mathcal{O}$ is a nonempty subset of Σ, then*

(i) ${}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_u}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_r}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_i}(\mathcal{O})$.
(ii) ${}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_u}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_l}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_i}(\mathcal{O})$.
(iii) ${}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda\langle u\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda\langle r\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda\langle i\rangle}(\mathcal{O})$.
(iv) ${}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda\langle u\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda\langle l\rangle}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda\langle i\rangle}(\mathcal{O})$.

Next, we will compare the approximations and accuracy criteria presented in this section, which are based on topological structures, with the analogous methods introduced in the previous section.

Proposition 19 *Consider* $\mathrm{R} \vee \mathrm{T}$ *is an ideal on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$. If $\mathcal{O} \subseteq \Sigma$, then*

(i) $\underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$*, for each* ξ.
(ii) $\overline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_\xi}(\mathcal{O}) \supseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_\xi}(\mathcal{O})$*, for each* ξ.

Proof To prove (i). Let $\sigma_1 \in \underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_\xi}(\mathcal{O})$, then there exists a subset $V \in {}^{\mathrm{R}\vee\mathrm{T}}\Omega_\xi$ such that $\sigma_1 \in V \subseteq \mathcal{O}$. This implies that $\mathbb{E}_\xi(\sigma_1)\backslash V \in \mathrm{R} \vee \mathrm{T}$. Since $V \subseteq \mathcal{O}$, then $\mathbb{E}_\xi(\sigma_1)\backslash\mathcal{O} \in \mathrm{R} \vee \mathrm{T}$. Hence, $\sigma_1 \in {}^{\mathrm{R}\vee\mathrm{T}}\widetilde{\mathcal{F}}_{\mathbb{E}_\xi}(\mathcal{O})$. Since $\sigma_1 \in \mathcal{O}$, then $\sigma_1 \in {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$ and so $\underline{{}^{\mathrm{R}\vee\mathrm{T}}\lambda_\xi}(\mathcal{O}) \subseteq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_\xi}(\mathcal{O})$.

By the same manner, one can prove the rest items of this proposition. □

Corollary 10 *Consider* $\mathrm{R} \vee \mathrm{T}$ *is an ideal on an ξ-NS $(\Sigma, \rho, \varrho_\xi)$. If $\mathcal{O} \subseteq \Sigma$, then* ${}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_\xi}(\mathcal{O}) \leq {}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O})$*, for each* ξ.

The converse of Corollary 10 need not to be true, refer to Example 6. Suppose that $\sigma = \langle u\rangle$ and $\mathcal{O} = \{\sigma_1, \sigma_4\}$. Then ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) = \frac{1}{3}$, ${}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_\xi}(\mathcal{O}) = \frac{1}{4}$. Hence, ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}_\xi}(\mathcal{O}) \nleq {}^{\mathrm{R}\vee\mathrm{T}}\varphi_{\lambda_\xi}(\mathcal{O})$.

Table 2 Patients with various symptoms and report

Patients	Rashes	Fever	Headache	Vomiting	Fatigue	Decision
σ_1	+	+	-	-	+	✓
σ_2	-	+	+	+	+	✓
σ_3	+	+	-	+	-	✓
σ_4	-	-	+	+	-	✗
σ_5	+	+	-	-	+	✗
σ_6	+	+	-	+	+	✗
σ_7	+	+	-	+	-	✓
σ_8	+	-	-	-	-	✗

5 An application of the proposed approach to diagnosis of dengue disease

This section examines our models' efficiency in dealing with dengue disease information systems for some patients. We demonstrate how our approach assists in improving the made decision and how we benefit from a topological technique to assign the most important symptoms in deciding the decision. According to the below analysis, we can say that our proposed paradigms outperform the existing paradigms. In contrast, we point out the limitation(s) induced by the approach introduced in Subsection 3.1.

In Table 2, we display a set of eight patients $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\}$ with the following symptoms (known in the information system as conditional attributes): rashes, fever, headache, vomiting, and fatigue. Whereas the dengue report is the decision attribute. For each symptom (conditional attribute), we give one of the values + or - depending on whether the patient has that symptom or not. Also, we give one of the values ✓, or ✗ for the decision attribute.

Let us consider the expert define a relation ρ on Σ as follows

$$\sigma_i \rho \sigma_j \iff \sigma_i \text{ and } \sigma_j \text{ have similar positive symptoms greater than two}$$

Then $\rho = \{(\sigma_1, \sigma_1), (\sigma_2, \sigma_2), (\sigma_3, \sigma_3), (\sigma_5, \sigma_5), (\sigma_6, \sigma_6), (\sigma_7, \sigma_7), (\sigma_1, \sigma_5), (\sigma_5, \sigma_1), (\sigma_1, \sigma_6), (\sigma_6, \sigma_1), (\sigma_2, \sigma_6),$
$(\sigma_6, \sigma_2), (\sigma_3, \sigma_6), (\sigma_6, \sigma_3), (\sigma_3, \sigma_7), (\sigma_7, \sigma_3), (\sigma_5, \sigma_6), (\sigma_6, \sigma_5), (\sigma_6, \sigma_7), (\sigma_7, \sigma_6)\}$.

It can be seen that the given relation ρ is symmetric but it is neither reflexive (since $(\sigma_4, \sigma_4\}) \notin \rho$) nor transitive (since $(\sigma_2, \sigma_5\}) \notin \rho$ in spite of being $(\sigma_2, \sigma_6\}), (\sigma_6, \sigma_5\}) \in \rho$). To process these data, we should initiate $\mathbb{E}_\xi$-neighbourhoods systems. According to the symmetric characteristic of this relation, we obtain that all $\mathbb{E}_\xi$-neighbourhoods are equal (by Proposition 2). In Table 3, we provide $\mathbb{E}_\xi$ for each patient.

Also, let $\mathrm{R} = \{\emptyset, \{\sigma_4\}, \{\sigma_5\}, \{\sigma_4, \sigma_5\}\}$, $\mathrm{T} = \{\emptyset, \{\sigma_7\}\}$ refer to the ideals given by two experts. Hence, $\mathrm{R} \cup \mathrm{T} = \{\emptyset, \{\sigma_4\}, \{\sigma_5\}, \{\sigma_7\}, \{\sigma_4, \sigma_5\}\}$, $\mathrm{R} \vee \mathrm{T} = \{\emptyset, \{\sigma_4\}, \{\sigma_5\}, \{\sigma_7\}, \{\sigma_4, \sigma_5\}, \{\sigma_4, \sigma_7\}, \{\sigma_5, \sigma_7\}, \{\sigma_4, \sigma_5, \sigma_7\}\}$. We compute the approximation operators and accuracy measures for a set of patients they have positive report

Table 3 $\mathbb{E}_{\xi}$ for each patient

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8
$\mathcal{G}_r()$	$\{\sigma_1, \sigma_5, \sigma_6\}$	$\{\sigma_2, \sigma_6\}$	$\{\sigma_3, \sigma_6, \sigma_7\}$	$\emptyset$	$\{\sigma_1, \sigma_5, \sigma_6\}$	$\{\sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_6, \sigma_7\}$	$\{\sigma_3, \sigma_6, \sigma_7\}$	$\emptyset$
$\mathcal{G}_{\langle r\rangle}()$	$\{\sigma_1, \sigma_5, \sigma_6\}$	$\{\sigma_2, \sigma_6\}$	$\{\sigma_3, \sigma_6, \sigma_7\}$	$\emptyset$	$\{\sigma_1, \sigma_5, \sigma_6\}$	$\{\sigma_6\}$	$\{\sigma_3, \sigma_6, \sigma_7\}$	$\emptyset$
$\mathbb{E}_r()$	$\{\sigma_1, \sigma_3, \sigma_5, \sigma_7\}$	$\{\sigma_2\}$	$\{\sigma_1, \sigma_3, \sigma_5, \sigma_7\}$	$\{\sigma_4, \sigma_8\}$	$\{\sigma_1, \sigma_3, \sigma_5, \sigma_7\}$	$\{\sigma_6\}$	$\{\sigma_1, \sigma_3, \sigma_5, \sigma_7\}$	$\{\sigma_4, \sigma_8\}$

of dengue $\mathcal{H} = \{\sigma_1, \sigma_3, \sigma_4, \sigma_6\}$ by using models introduced in [14, 15] and our models given herein as follows.

- Approach introduced in [14].

  (i) $\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{H}) = \{\sigma_6\}$,
  (ii) $\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{H}) = \{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\}$,
  (iii) $\mathbb{A}_{\mathbb{E}_{\xi}}(\mathcal{H}) = \frac{|\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{H})|}{|\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{H})|} = \frac{1}{7}$.

- Approach introduced in [15].

  (i) ${}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{H}) = \{\sigma_6\}$,
  (ii) ${}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{H}) = \{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$,
  (iii) ${}^{\mathrm{R}}\mathcal{A}_{\mathbb{E}_{\xi}}(\mathcal{H}) = \frac{|{}^{\mathrm{R}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{H})|}{|{}^{\mathrm{R}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{H})|} = \frac{1}{6}$.
  (iv) ${}^{\mathrm{T}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{H}) = \{\sigma_6\}$ and
  (v) ${}^{\mathrm{T}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{H}) = \{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\}$,
  (vi) ${}^{\mathrm{T}}\mathcal{A}_{\mathbb{E}_{\xi}}(\mathcal{H}) = \frac{|{}^{\mathrm{T}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{H})|}{|{}^{\mathrm{T}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{H})|} = \frac{1}{7}$.

- Approach introduced herein (in Subsection 3.2).

  (i) ${}^{\mathrm{R}\cup\mathrm{T}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{H}) = \{\sigma_6\}$,
  (ii) ${}^{\mathrm{R}\cup\mathrm{T}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{H}) = \{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$,
  (iii) ${}^{\mathrm{R}\cup\mathrm{T}}\mathcal{A}_{\mathbb{E}_{\xi}}(\mathcal{H}) = \frac{|{}^{\mathrm{R}\cup\mathrm{T}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{H})|}{|{}^{\mathrm{R}\cup\mathrm{T}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{H})|} = \frac{1}{6}$.

- Approach introduced herein (in Subsection 3.4)

  (i) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{H}) = \{\sigma_1, \sigma_3, \sigma_6\}$,
  (ii) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{H}) = \{\sigma_1, \sigma_3, \sigma_4, \sigma_6, \sigma_7\}$,
  (iii) ${}^{\mathrm{R}\vee\mathrm{T}}\mathcal{A}_{\mathbb{E}_{\xi}}(\mathcal{H}) = \frac{|{}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}_{\mathbb{E}_{\xi}}(\mathcal{H})|}{|{}^{\mathrm{R}\vee\mathrm{T}}\mathcal{F}^{\mathbb{E}_{\xi}}(\mathcal{H})|} = \frac{3}{5}$.

- A topological approach presented in Sect. 4. First, the structure of topology produced by Table 3 is ${}^{\mathrm{R}\vee\mathrm{T}}\Omega_{\mathbb{E}_{\xi}} = \{\emptyset, \Sigma, \{\sigma_2\}, \{\sigma_6\}, \{\sigma_8\}, \{\sigma_2, \sigma_6\}, \{\sigma_2, \sigma_8\}, \{\sigma_4, \sigma_8\}, \{\sigma_6, \sigma_8\}, \{\sigma_2, \sigma_4, \sigma_8\}, \{\sigma_2, \sigma_6, \sigma_8\}, \{\sigma_4, \sigma_6, \sigma_8\}, \{\sigma_2, \sigma_4, \sigma_6, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_6\}, \{\sigma_1, \sigma_3, \sigma_7\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_7\}, \{\sigma_1, \sigma_3, \sigma_6, \sigma_7\}, \{\sigma_1, \sigma_3, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_5, \sigma_7\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_7\}, \{\sigma_1, \sigma_3, \sigma_5, \sigma_6, \sigma_7\}, \{\sigma_1, \sigma_3, \sigma_5, \sigma_7, \sigma_8\}, \{\sigma_1,$

$\sigma_2, \sigma_3, \sigma_5, \sigma_6, \sigma_7\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_6, \sigma_7\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_4, \sigma_6, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_4, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_5, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6, \sigma_7, \sigma_8\}\}$.

(i) $\underline{{}^{R\vee T}\lambda_\xi}(\mathcal{H}) = {}^{R\vee T}int_{\mathbb{E}_\xi}(\mathcal{H}) = \{\sigma_1, \sigma_3, \sigma_6\}$,

(ii) $\overline{{}^{R\vee T}\lambda_\xi}(\mathcal{H}) = {}^{R\vee T}cl_{\mathbb{E}_\xi}(\mathcal{H}) = \{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$, and

(iii) ${}^{R\vee T}\varphi_{\lambda_\xi}(\mathcal{H}) = \frac{|\underline{{}^{R\vee T}\lambda_\xi}(\mathcal{H})|}{|\overline{{}^{R\vee T}\lambda_\xi}(\mathcal{H})|} = \frac{1}{2}$.

According to the above, one can see that these computations are in agreement with the results obtained in Proposition 12 and Corollary 5. This means that the best development for the lower approximations and upper approximations, and accuracy measures of subsets compared to other rough set models presented herein and those displayed in [14, 15] is produced by the models introduced in Subsection 3.4. Furthermore, we use the topological paradigms of rough sets defined in Sect. 4 to calculate the approximations (lower and upper) and boundary regions of subsets. Because of the failures of rough set models established in Subsection 3.1 and Subsection 3.3 about accuracy measures and preserving the main characteristics of the standard models, we see it is appropriate to neglect these models in this part.

In the remaining part of this Section, we apply a topology ${}^{R\vee T}\Omega_{\mathbb{E}\xi}$ (that we generate above by cardinality neighborhoods and ideals) to determine the key symptoms to judge whether the patient infected by dengue disease or not, where ${}^{R\vee T}\Omega_{\mathbb{E}\xi} = \{\emptyset, \Sigma, \{\sigma_2\}, \{\sigma_6\}, \{\sigma_8\}, \{\sigma_2, \sigma_6\}, \{\sigma_2, \sigma_8\}, \{\sigma_4, \sigma_8\}, \{\sigma_6, \sigma_8\}, \{\sigma_2, \sigma_4, \sigma_8\}, \{\sigma_2, \sigma_6, \sigma_8\}, \{\sigma_4, \sigma_6, \sigma_8\}, \{\sigma_2, \sigma_4, \sigma_6, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_6\}, \{\sigma_1, \sigma_3, \sigma_7\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_7\}, \{\sigma_1, \sigma_3, \sigma_6, \sigma_7\}, \{\sigma_1, \sigma_3, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_5, \sigma_7\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_7\}, \{\sigma_1, \sigma_3, \sigma_5, \sigma_6, \sigma_7\}, \{\sigma_1, \sigma_3, \sigma_5, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_6, \sigma_7\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_6, \sigma_7\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_4, \sigma_6, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_4, \sigma_7, \sigma_8\}, \{\sigma_1, \sigma_3, \sigma_5, \sigma_8\}, \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6, \sigma_7, \sigma_8\}\}$.

(i) If we neglect the symptom "rashes" from the conditional attributes, then $\rho_{rashes} = \{(\sigma_2, \sigma_2), (\sigma_6, \sigma_6), (\sigma_2, \sigma_6), (\sigma_6, \sigma_2)\}$. It is clear that ${}^{R\vee T}\Omega_{\mathbb{E}\xi} - rashes \neq {}^{R\vee T}\Omega_{\mathbb{E}\xi}$.

(ii) If we exclude the symptom "fever" from the conditional attributes, then one can check that ${}^{R\vee T}\Omega_{\mathbb{E}\xi} - fever \neq {}^{R\vee T}\Omega_{\mathbb{E}\xi}$.

(iii) If we remove the symptom "headache" from the conditional attributes, then we obtain a similar relation ρ, which implies that ${}^{R\vee T}\Omega_{\mathbb{E}\xi} - headache = {}^{R\vee T}\Omega_{\mathbb{E}\xi}$.

(iv) If we cancel the symptom "vomiting" from the conditional attributes, then one can check that ${}^{R\vee T}\Omega_{\mathbb{E}\xi} - vomiting \neq {}^{R\vee T}\Omega_{\mathbb{E}\xi}$.

(v) If we omit the symptom "fatigue" from the conditional attributes, then one can check that ${}^{R\vee T}\Omega_{\mathbb{E}\xi} - fatigue \neq {}^{R\vee T}\Omega_{\mathbb{E}\xi}$.

According to the above computations, we get that the symptoms of rashes, fever, vomiting, and fatigue are the core of attributes; i.e., we arrive at the conclusion that rashes, fever, vomiting, and fatigue are the key symptoms to judge whether the patient has dengue disease or not.

6 Discussions: strengths and limitations

In what follows, we reason why we need the current rough set models. We show their advantages and how they expand the domain of application. Also, we refer to some remarks that should be taken into account when we apply these models to avoid shortcomings. Moreover, we compare the four proposed rough set models in terms of accuracy measures and their ability to maintain the properties of Pawlak's model.

– Advantages

(i) The present rough set models are constructed by any arbitrary relation, which widens the situations that are covered by these models compared to the standard models of Pawlak and others generated by equivalence relations or some specific kinds of relations such as those models introduced in [14, 15].
(ii) The rough set models introduced herein provide an efficient instrument to address some issues that consider the number of elements associated with each other under arbitrary relations. They also enhance the previous rough paradigms by using two ideals instead of one since the use of two ideals provides two perspectives rather than just one. This hybridized technique reduces the boundary region and increases the degree of accuracy compared to [14, 15], which is in agreement with the primary objective of rough set theory.
(iii) Rough sets models presented in Subsection 3.2 and Subsection 3.4 maintain most properties of the standard models of Pawlak (given in Proposition 1) as clarified in Proposition 9 and Proposition 9. These two models keep the improvements obtained by other models introduced in Subsection 3.1 and Subsection 3.3 in relation to enhancing the lower approximation and shrinking the upper approximation, moreover, they eliminate the defects resulting from the first and third models.
(iv) We initiate a counterpart topological model for the fourth model (the best one) to assist numerous users with abstract backgrounds such as topologists in selecting the methods appropriate to their experiences. These sorts of users prefer to cope with the topological techniques because of the ease of computing the approximation operators from their corresponding interior and closure topological operators.

– Limitations

(i) Our models displayed in Subsection 3.1 and Subsection 3.3 lose some standard model properties, such as the property describing the relationship between the subsets and their lower and upper approximations. This means that the relationship used to calculate the accuracy measures in these two models should be updated; it cannot be applied directly, as this will otherwise lead to contradictions.
(ii) The proposed models do not satisfy the monotonicity property since the cardinal rough neighborhoods change in a way that cannot be determined when we maximize or minimize the given relation.

– Comparisons
According to the computations presented in the previous section, the fourth model is the most preferred as it yields the highest accuracy and preserves the greatest number of properties from Pawlak's standard model. The second model follows in terms of maintaining these two characteristics. In contrast, the first and third models encounter issues with calculating accuracy measures and lose many of the fundamental properties of Pawlak's standard model.

7 Conclusion and future work

Rough set theory is one of the most popular and powerful methods to cope with uncertainty and high dimensionality of a wealth of data, which pose challenges to the fields of data mining, pattern recognition, and computational intelligence. One core merit of rough set theory is its ability to represent data using a granular structure without requiring prior information beyond the dataset. Traditionally, this granular structure, represented by equivalence classes, has been refined using neighborhood systems inspired by arbitrary relations, which relaxes the strict conditions of equivalence relations.

In this article, we have adopted novel types of generalized approximation spaces utilizing the concepts of cardinality neighborhoods and ideals. These types of rough paradigms differ from the previous types in two ways: the first is that two ideals are used instead of a single ideal, and the second is the method of forming a new structure from these ideals, as we used two methods for the construction as follows:

$$\text{union construction: } \mathrm{R} \cup \mathrm{T} = \{\mathcal{A} : \mathcal{A} \in \mathrm{R} \text{ or } \mathcal{A} \in \mathrm{T}\} \text{ and}$$
$$\text{joint construction: } \mathrm{R} \vee \mathrm{T} = \{\mathcal{A} \cup \mathcal{B} : \mathcal{A} \in \mathrm{R}, \mathcal{B} \in \mathrm{T}\}$$

Since the structure $\mathrm{R} \vee \mathrm{T}$ contains $\mathrm{R} \cup \mathrm{T}$, we find that the generalized approximation spaces inspired by $\mathrm{R} \vee \mathrm{T}$ expand the domain of confirmed knowledge more effectively than those inspired by $\mathrm{R} \cup \mathrm{T}$. Moreover, while the structure $\mathrm{R} \vee \mathrm{T}$ constitutes an ideal, $\mathrm{R} \cup \mathrm{T}$ does not, leading to the dissipation of some properties of Pawlak's standard model in the generalized approximation spaces inspired by $\mathrm{R} \cup \mathrm{T}$.

We have explored the key properties of the four rough set models introduced in Sect. 3 and demonstrated that the second and fourth models are preferable for analyzing information systems. In contrast, we showed that the first and third models have limitations concerning maintaining the main features of the original paradigm, and some formulas used to calculate accuracy are invalid in certain cases. Then, we constructed topological spaces using joint ideals $\mathrm{R} \vee \mathrm{T}$ as an analogy for the fourth proposed rough paradigm. We have illustrated its relationships with its counterpart paradigm set up in Subsection 3.4. In this regard, the space constructed by union ideals $\mathrm{R} \cup \mathrm{T}$ fails to be a topology; it institutes a supra topology, so we delay the investigation of this structure for future works. We have examined the performance and efficiency of the proposed paradigms via a medical example of dengue disease; the outcomes of our computations indicate that the rough set models adopted herein outperform existing models. Last but not least, we have listed the merits of the proposed models and identified their limitations.

In the future, we intend to achieve the following studies:

– Integrate the technique of a finite set of ideals with the other types of existing rough neighborhoods to establish fresh rough set models.
– Develop the proposed rough set models using a finite set of arbitrary relations instead of one. This hybridization will significantly increase the accuracy.
– look at these models within the framework of soft rough set settings.

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Declarations

Conflict of interest The authors declare that there is no Conflict of interest regarding the publication of this article.

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